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LAMINAR NATURAL CONVECTION FLOW IN MAGNETO-HYDRODYNAMICS

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Abstract—This paper deals with the two-dimensional laminar natural convection flow of an electrically conducting viscous fluid, such as mercury or liquid sodium, in the presence of electric or magnetic fields. Two different flow régimes are discussed. The first example considered is the steady fully developed natural convection flow, with and without heat sources, between two long parallel plane surfaces with uniform magnetic field applied normal to the surfaces. The plane vertical surfaces are open at both ends to the ambient fluid and are maintained at constant temperatures different from that of the ambient fluid. Tables are given from which the fully developed temperature, velocity and induced magnetic fields may be found. Flow characteristics such as the net mass flow and wall Nusselt numbers are also evaluated.

The second example considered is the steady two-dimensional natural convection flow set up by Joule heating when a direct current flows in the axial direction through a horizontal circular tube filled with an electrically conducting viscous fluid. The outside surface of the tube is maintained at constant temperature by a coolant which is assumed to be a non-conductor and non-magnetic. The influence of the non-uniform convection flow on the temperature distribution and wall Nusselt number is calculated.

Résumé—Cet article traite de l'écoulement de convection naturelle laminaire à deux dimensions dans un fluide visqueux conducteur tel que le mercure ou le sodium liquide, en présence de champs magnétiques ou électriques. Deux régimes d'écoulement différents sont étudiés. Le premier exemple considéré est celui de la convection naturelle, en régime permanent, avec ou sans source de chaleur, entre deux longues surfaces planes parallèles normalement auxquelles est appliqué un champ magnétique uniforme. Les surfaces planes verticales sont ouvertes aux deux extrémités vars le fluide ambiant et sont maintenues à des températures constantes, différentes de celle du fluide ambiant. Des tables, à partir desquelles on peut trouver la température de régime, la vitesse et les champs magnétiques induits, sont données. Des caractéristiques de l'écoulement, telles que le débit massique et le nombre de Nusselt à la paroi sont également évaluées.

Le deuxième exemple considéré est celui de la convection naturelle à deux dimensions qui s'établit dans le cas d'un chauffage par effet Joule, quand on fait circuler un courant continu dans l'axe d'un tube horizontal, à section circulaire, rempli d'un liquide visqueux conducteur. La surface extérieure du tube est maintenue à température constante par un réfrigérant que l'on suppose non conducteur et non magnétique. L'influence d'un écoulement de convection naturelle non-uniforme sur la distribution des températures et le nombre de Nusselt à la paroi est calculée.

Zusammenfassung—Die Arbeit behandelt die zweidimensionale, laminare, freie Konvektionsströmung in einer elektrisch leitenden, viskosen Flüssigkeit, wie Quecksilber oder flüssigem Natrium in Gegenwart von elektrischen oder magnetischen Feldern. Zwei verschiedene Strömungsarten werden besprochen: Als erstes Beispiel die stationäre, voll ausgebildete, freie Konvektionsströmung mit und ohne Wärmequellen zwischen zwei langen parallelen ebenen Flächen mit gleichmässigem, senkrecht zu den Oberflächen wirkendem Magnetfeld. Der Spalt zwischen den ebenen, senkrecht stehenden Oberflächen ist für die umgebende Flüssigkeit an beiden Enden geöffnet. Die Oberflächen werden auf konstanten Temperaturen gehalten, die sich von der Temperatur der angrenzenden Flüssigkeit unterscheiden. Aus angegebenen Tabellen können Temperatur-, Geschwindigkeits- und induzierte Magnetfelder ersehen werden. Weiterhin sind charakteristische Strömungsgrössen wie Mengenstrom und Nusseltzahl in Wandnähe berechnet.

Als zweites Beispiel wird die stationäre zweidimensionale freie Konvektionsströmung betrachtet, die infolge Joulescher Erwärmung auftritt, wenn Gleichstrom in Achsialrichtung durch ein waagerechtes, mit elektrisch leitender viskoser Flüssigkeit gefülltes Rohr von Kreisquerschnitt fliesst. Die Rohraussenfläche wird durch ein nichtleitendes und unmagnetisches Kühlmittel auf konstanter Temperatur gehalten. Der Einfluss des ungleichförmigen Konvektionsstromes auf die Temperaturverteilung und die Nusseltzahl in Wandnähe ist angegeben.

Аннотация—В статье рассматривается двухмерный ламинарный конвективный поток в условиях естественного движения электрически проводящей вязкой жидкости. Например, ртутр или натрий в жидком состоянии. Предусматривается наличие электрических или магнитных полей. Рассматриваются два раэличных режима потока. В первом случае—стационарный полностью установившейся поток в условиях естественной конвекции между двумя длинными плоско-параллельными поверхностями при наличии источников тепла или без них и с равномерным магнитным полем, перпендикулярным к поверхности пластины. Температура плоских вертикальных поверхностей отлична от температуры окружающей жидкости, с которой соприкасаются оба конца пластины. Из приведенных таблиц можно найти полностью установившиеся поля температуры, скорости, а также индуктировнные магнитные поля, Вычислены характеристики чистого потока массы и число Нуссельта стенки.

Во втором случае рассматривается стационарный двухмерный поток в условиях естественной конвекции, создаваемый нагреванием при прохождении постоянного тока в аксиальном направлении через горизонтальную круглую трубу, заполненную электрически проводящей вязкой жидкостю. Температура внешней поверхности трубы поддерживается постоянной при помощи охлаждающей среды, которую принимают непроводящей и немагнитной. Вычислено влияние неравномерного конвективного потока на распределние температуры и на число Нуссельта стенки.

NOMENCLATURE

e.m.u. and c.g.s. system used throughout.

- $\mathbf{A}$ dimensionless magnetic vector potential;
- a characteristic length;
- c_p specific heat at constant pressure;
- $\mathbf{E}$ electrical intensity;
- $\mathbf{g}$ gravitational force per unit of mass;
- G Grashof number, $\frac{\beta g \theta_w a^3}{\nu^2}$ (Example I) or $\frac{\beta g J_0^2 a^5}{k \sigma \nu^2}$ (Example II);
- G_M magnetic Grashof number, $4\pi\sigma\mu_e\nu G$;
- $\mathbf{H}$ magnetic intensity;
- $\mathbf{J}$ current density;
- K dimensionless parameter, PGK_A;
- K_A dimensionless parameter $\frac{\beta g a}{c_p}$;
- k thermometric conductivity;
- M Hartmann number, $\mu_e H_0 a \sqrt{\frac{\sigma}{\rho\nu}}$;
- Nu Nusselt number;
- p pressure;
- p_d pressure p minus the hydrostatic pressure p_s;
- P Prandtl number $\frac{\rho c_p \nu}{k}$;
- Q heat added by heat sources;
- $\mathbf{q}$ fluid velocity;
- T temperature;
- x, y, z Cartesian co-ordinates;
- r, ϕ, z cylindrical co-ordinates.

Greek symbols

- α dimensionless heat source parameter;
- β coefficient of volumetric expansion;
- γ dimensionless quantity, $\frac{\mu_{1e}}{\mu_{2e}}$;
- Θ dimensionless local temperature distribution;
- θ temperature difference, $T - T_s$;
- λ ratio of wall temperature differences, $\frac{\theta_{w1}}{\theta_{w0}}$;
- μ dynamic viscosity;
- μ_e magnetic permeability;
- ν kinematic viscosity;
- η magnetic viscosity;
- ρ fluid density;
- σ electrical conductivity;
- χ dimensionless group, $1/4\pi\sigma\mu_{1e}\nu$;
- ϵ dimensionless group, $1/PK_A\chi$;
- Ψ dimensionless stream function;
- Φ magnetic scalar potential.

Subscripts

- s reference condition (usually taken as the hydrostatic condition);
- w wall condition;
- 1 and 2 wall condition at $y = 0$ and a respectively for Example I, and conditions in the fluid and coolant respectively for Example II.

1. INTRODUCTION AND BASIC THEORY OF TWO-DIMENSIONAL NATURAL CONVECTION IN MAGNETO-HYDRODYNAMICS

THE PROCESS of natural convection flow will occur when density variations due to heating exist in a fluid, and is generated entirely by the action of body forces due to gravity. This process has many applications as a mechanism for heat transfer and, for example, plays a major role in the cooling of nuclear power plants where liquid sodium is used as a coolant. It is the purpose of this paper to investigate some aspects of two-dimensional laminar natural convection flow of an electrically conducting viscous fluid, such as mercury or liquid sodium, in the presence of additional forces due to imposed electric and magnetic fields. In general two types of problem require investigation. In the first type if a natural convection flow exists we wish to know how this is modified, i.e. changes in heat transfer and net mass flow, when electric and magnetic fields are applied. The second type of problem which may occur is one in which the natural convection flow is a direct consequence of the applied electric and magnetic fields and would not exist in the absence of these fields. In this case temperature gradients are caused by Joule heating if the region under consideration is enclosed or partially enclosed by solid boundaries.

The paper treats two relatively simple two-dimensional natural convection flow régimes, representative of the above types. In the first example results are obtained for the steady fully developed natural convection flow, with and without heat sources, between two long parallel plane surfaces with uniform magnetic fields applied normal to the surfaces. The plane vertical surfaces are open at both ends to the ambient fluid and are maintained at constant temperatures different from that of the ambient fluid. This configuration is a modification of the magnetic field-free case as discussed by Ostrach [1] and the analysis of this example follows Ostrach's treatment of the thermal convection problem and that of Cowling [2] in the discussion of the Hartmann–Lazarus flow of an electrically conducting viscous fluid between parallel flat plates. The special case when the arithmetic average of the plate temperatures is equal to the temperature of the ambient fluid has been discussed by Gershuni and Zhukhovitskii* [3]. In this reference the viscous and Joulean dissipation have been neglected in the equation of thermal energy transport.

In the second example results are obtained for the natural convection flow régime set up by Joule heating when a direct current is passed axially through a horizontal cylindrical tube filled with conducting fluid. The outside surface of the tube is maintained at constant temperature by a coolant which is assumed to be a non-conductor and non-magnetic. End effects near the electrodes are neglected, so that it is assumed that the tube is sufficiently long to allow steady two-dimensional natural convection flow near the centre to be established.

Consider now the magneto-hydrodynamic equations and the equation of thermal energy transport relating to steady natural convection flow. The usual assumptions are made with regard to the physical properties of the fluid, i.e. the viscosity, electrical conductivity and magnetic permeability etc. are independent of the temperature and the strength of the magnetic or electric fields involved. The density of the fluid is assumed constant except in the case of density variation with temperature in producing the buoyancy force. The basic equations in e.m.u. and c.g.s. units are, in the usual notation,

$$\operatorname{div} \mathbf{q} = 0, \tag{1.1}$$

$$(\mathbf{q}.\nabla)\mathbf{q} = -1/\rho\,\nabla p_d + \nu\nabla^2 q + \frac{\mu_e}{\rho}\mathbf{J}\wedge\mathbf{H} - \beta(T - T_s)\mathbf{g}, \tag{1.2}$$

$$\rho c_p\,\mathbf{q}.\nabla T = k\nabla^2 T + \frac{\mathbf{J}^2}{\sigma} + Q + \Phi_q, \tag{1.3}$$

together with the electro-magnetic equations

$$\operatorname{div} \mathbf{H} = 0, \tag{1.4}$$

$$\operatorname{curl} \mathbf{H} = 4\pi J, \tag{1.5}$$

$$\operatorname{curl} \mathbf{E} = 0, \tag{1.6}$$

and finally Ohm's law for the moving fluid

$$\mathbf{J} = \sigma(\mathbf{E} + \mu_e\,\mathbf{q}\wedge\mathbf{H}). \tag{1.7}$$

* The author is indebted to a referee for this reference.

Here p_d is the pressure of the fluid minus the hydrostatic pressure, β is the coefficient of volumetric expansion, Q is the quantity of heat added by heat sources per unit of volume, and Φ_q is the viscous dissipation function. Furthermore equation (1.3) is valid only when the temperature difference $(T-T_s)$ is small compared with the hydrostatic temperature T_s, and all physical constants appearing in equations (1.1) to (1.7) must be evaluated at the hydrostatic condition.

These equations can now be reduced to their simplest two-dimensional form when the following conditions are satisfied within the fluid:

(i) the pressure gradient in the z-direction is zero i.e. $(\partial p_d/\partial z) = 0$,

(ii) the velocity and magnetic field components are

$$\mathbf{q} = [u(x, y), \quad v(x, y), 0], \tag{1.8}$$

and

$$\mathbf{H} = [H_x(x, y), H_y(x, y), 0]. \tag{1.9}$$

On using this assumed form for $\mathbf{H}$ equation (1.5) implies that $J_x = J_y = 0$ and

$$4\pi J_z = \frac{\partial H_y}{\partial x} = \frac{\partial H_z}{\partial y};$$

equation (1.6) implies that

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0$$

and $E_z = E_0$, a constant; equation (1.7) now implies on using (1.8) and (1.9) that $E_x = E_y = 0$ and $J_z = \sigma[E_0 + \mu_e(uH_y - vH_x)]$. Thus subject to the above conditions the equations governing the steady two-dimensional natural convection flow, taking the x-axis in the vertical direction, are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1.10}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p_d}{\partial x} + \nu\nabla^2 u - \frac{\mu_e}{\rho} J_z H_y + \beta g(T - T_s), \tag{1.11}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p_d}{\partial y} + \nu\nabla^2 v + \frac{\mu_e}{\rho} J_z H_x. \tag{1.12}$$

$$\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} = 0, \tag{1.13}$$

$$J_z = \frac{1}{4\pi}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) = \sigma[E_0 + \mu_e(uH_y - vH_x)]. \tag{1.14}$$

and

$$\rho c_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = k\nabla^2 T + \frac{J_z^2}{\sigma} + Q + \Phi_q, \tag{1.15}$$

where Φ_q is the two-dimensional form of the dissipation function

$$\Phi_q = 2\mu\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2\right]. \tag{1.16}$$

The boundary conditions for equations (1.10) and (1.16) will be derived later.

2. EXAMPLE I: FULLY DEVELOPED NATURAL CONVECTION FLOW BETWEEN PARALLEL FLAT PLATES IN A MAGNETIC FIELD

2(a) *Statement of problem and governing equations*

The simplified configuration to be investigated in this section is the fully developed laminar flow of an electrically conducting viscous fluid, with and without heat sources, between two parallel flat plates orientated in the vertical direction (taken to be the x-direction) and distance a apart (i.e. $y = 0$ and a). The plate surfaces, which are open at both ends to the ambient fluid, are maintained at constant temperature $T = T_0$ at $y = 0$ and $T = T_1$ at $y = a$ respectively, and in general $T_0 \neq T_1 \neq T_s$, the ambient temperature. A uniform magnetic field of intensity H_0 is applied normal to the plates, i.e. parallel to Oy and perpendicular to

Ox and Oz respectively. If the dimensions of the plates are large compared with the distance between them such that fully developed flow exists everywhere (except near the edges of the plates) then as in the Hartmann–Lazarus flow (see Cowling [2]) a solution of the basic equations (1.10) to (1.16) is possible provided $u = u(y)$, $v = 0$, $T = T(y)$, $H_x = H_x(y)$ and $H_y = H_0$, the applied field. The equations for steady motion now become

$$\mu \frac{d^2u}{dy^2} + \beta\rho g(T - T_s) - \mu_e H_0 J_z = \frac{\partial p_d}{\partial x}, \quad (2.1)$$

$$\mu_e J_z H_x = \frac{\partial p_d}{\partial y}, \quad (2.2)$$

$$J_z = -\frac{1}{4\pi}\frac{dH_x}{dy} = \sigma(E_0 + \mu_e H_0 u), \quad (2.3)$$

and

$$k\frac{d^2T}{dy^2} + \mu\left(\frac{du}{dy}\right)^2 + Q + \sigma(E_0 + \mu_e H_0 u)^2 = 0. \quad (2.4)$$

Now we are interested only in the case for which there is zero axial pressure gradient i.e. $\partial p_d/\partial x = 0$ and thus on using equations (2.2) and (2.3) the momentum equation yields

$$\mu \frac{d^2u}{dy^2} + \beta\rho g(T - T_s) - \sigma\mu_e H_0(E_0 + \mu_e H_0 u) = 0. \quad (2.5)$$

The boundary conditions associated with (2.4) and (2.5) are

$$\left.\begin{aligned} u(0) = u(a) = 0,\ T(0) = T_0 \\ \text{and} \qquad T(a) = T_1. \end{aligned}\right\} \quad (2.6)$$

A solution of equations (2.4) to (2.6) may now be obtained for any specified value of the constant electric field E_0 and this would, on using equation (2.3), imply a flow of current in the z-direction. Suppose now the channel is bounded by electrically insulated walls in the planes $z = \pm d$, where $d \gg a$. For this model a non-uniform induced electric field is set up since J_z must vanish at $z = \pm d$. Equations (2.4) and (2.5) no longer apply to this situation since the condition that E_0 must be constant is violated. To represent this model as closely as possible Cowling [2] suggests that the constant E_0 is adjusted to make the total current $\int_0^a J_z dy$ flowing between $z = \pm d$ vanish. This gives

$$E_0 = -\frac{\mu_e H_0}{a}\int_0^a u\, dy \quad (2.7)$$

which is consistent with the boundary condition $H_x(0) = H_x(a) = 0$. This follows from the condition that there must be no discontinuity in the tangential component of **H** at the solid interfaces $y = 0, a$. Thus for a long rectangular duct, having aspect ratio $2d/a \gg 1$, orientated in the vertical direction and with a uniform magnetic field imposed perpendicular to the isothermal walls $y = 0$, a, the steady fully developed natural convection flow near the central axis of the tube ($z = 0$) and sufficiently far away from the open ends, is determined by the equations

$$\mu \frac{d^2u}{dy^2} + \rho\beta g(T - T_s) - \sigma\mu_e^2 H_0^2\left(u - \frac{1}{a}\int_0^a u dy\right) = 0, \quad (2.8)$$

$$k\frac{d^2T}{dy^2} + \mu\left(\frac{du}{dy}\right)^2 + Q + \sigma\mu_e^2 H_0^2\left(u - \frac{1}{a}\int_0^a u dy\right)^2 = 0, \quad (2.9)$$

subject to the boundary conditions (2.6). The terms in (2.8) denote the viscous, buoyancy and Lorentz forces respectively; the terms in (2.9) denote the transport of thermal energy by conduction, viscous dissipation, heat generation by heat sources, and Joulean dissipation. We note also that the Lorentz force opposes the buoyancy force, which leads to an effective decrease in velocity or net mass flow as H_0 increases.

As the units of field strength, temperature and velocity we choose H_0, $\theta_0 = (T_0 - T_s)$ and ν/a respectively. Dimensional analysis then leads to the introduction of four dimensionless parameters: the temperature difference ratio

$$\lambda = \frac{(T_1 - T_s)}{(T_0 - T_s)} = \frac{\theta_1}{\theta_0};$$

the heat source parameter $\alpha = Qa^2/k\theta_0$; the Hartmann number $M = \mu_e H_0 a(\sigma/\rho\nu)^{1/2}$; and, due to Ostrach [1], the dimensionless group $K = PGK_A$ where $P = (\rho c_p \nu/k)$ is the Prandtl number, $G = (\beta g \theta_w a^3/\nu^2)$ the Grashof number ($\theta_w = \theta_0$ or θ_1), and the dimensionless group $K_A = (\beta g a/c_p)$. We now introduce the new dimensionless variables

$$Y = y/a, \quad \Theta = K\frac{(T_1 - T_s)}{(T_0 - T_s)}, \quad U = K_A P \frac{\nu}{a} u. \tag{2.10}$$

When these substitutions are made in the basic equations of motion (2.8) and (2.9) we obtain

$$U'' + \Theta - M^2(U - \int_0^1 U \mathrm{d}Y) = 0, \tag{2.11}$$

$$\Theta'' + (U')^2 + M^2(U - \int_0^1 U \mathrm{d}Y)^2 + \alpha K = 0, \tag{2.12}$$

subject to the boundary conditions

$$\left.\begin{aligned} &U(0) = U(1) = 0, \Theta(0) = K \\ \text{and} \quad &\Theta(1) = \lambda K. \end{aligned}\right\} \tag{2.13}$$

In equations (2.11) and (2.12) the prime denotes differentiation with respect to Y.

It remains now to choose representative values of α, λ, M and K applicable, for example, to liquid metals. Without loss of generality we shall take $\alpha = 0$, 10 and 100, $\lambda = -1(1)2$ and $M = 0, 2, 4$ and 10. However, as seen from Table 1, we are interested only in a range of small or moderately small values of K. For example, considering the convection flow of mercury in a gravitational field at room temperature (20°C), and taking $\theta_0 = T_0 - T_s = 5$°C and the plates 5 cm apart (i.e. $a = 5$), then $K = 1{\cdot}5$; for liquid sodium, if $T_s = 200$°C, $\theta_0 = 100$°C and $a = 10$ then $K = 1{\cdot}04$. In the next section series expansions in powers of K are obtained for U and Θ, valid for small K and all values of the remaining parameters α, λ and M.

2(b) *Series expansion for small K*

A series solution for U and Θ may be obtained by taking

Table 1

	Mercury, 20°C	Liquid sodium, 200°C
Density, ρ (g/cm^3)	13·55	0·904
Coefficient of cubical expansion, β (/°C)	$1{\cdot}82 \times 10^{-4}$	$2{\cdot}2 \times 10^{-4}$
Dynamic viscosity, μ (g/cm sec)	$1{\cdot}53 \times 10^{-2}$	$4{\cdot}5 \times 10^{-3}$
Kinematic viscosity, ν (cm^2/sec)	$1{\cdot}13 \times 10^{-3}$	$4{\cdot}97 \times 10^{-3}$
Permeability, μ_e (e.m.u.)	1	1
Resistivity, R (ohms-cm)	$9{\cdot}58 \times 10^{-5}$	$1{\cdot}36 \times 10^{-5}$
Specific heat, c_p (cal/°C)	$3{\cdot}33 \times 10^{-2}$	0·32
Thermal conductivity, K (cal/cm sec°C)	$1{\cdot}90 \times 10^{-2}$	0·195
Prandtl number, $P = \dfrac{\rho c_p \nu}{K}$	$2{\cdot}68 \times 10^{-2}$	$7{\cdot}38 \times 10^{-3}$
$\eta/\nu = \dfrac{1}{4\pi\sigma\mu_e\nu}$	$6{\cdot}74 \times 10^{6}$	$2{\cdot}18 \times 10^{5}$

$$\left.\begin{aligned} U &= K \sum_{n=0}^{\infty} K^n U_n(Y) \\ \text{and} \quad \Theta &= K \sum_{n=0}^{\infty} K_n \Theta_n(Y). \end{aligned}\right\} \tag{2.14}$$

On substituting the expansions (2.14) into equations (2.11) to (2.13) and equating coefficients of like powers of K, there results a set of differential equations of which the first few are:

$$\left.\begin{aligned} &\Theta_0'' + \alpha = 0, \Theta_0(0) = 1, \quad \Theta_0(1) = \lambda; \\ &U_0'' + \Theta_0 - M^2(U_0 - \textstyle\int_0^1 U_0 \mathrm{d}Y) = 0, \\ &\qquad U_0(0) = U_0(1) = 0; \end{aligned}\right\} \tag{2.15}$$

$$\left.\begin{aligned} &\Theta_1'' + (U_0')^2 + M^2(U_0 - \textstyle\int_0^1 U_0 \mathrm{d}Y)^2 = 0, \\ &\qquad \Theta_1(0) = \Theta_1(1) = 0; \\ &U_1'' + \Theta_1 - M^2(U_1 - \textstyle\int_0^1 U_1 \mathrm{d}Y) = 0, \\ &\qquad U_1(0) = U_1(1) = 0. \end{aligned}\right\} \tag{2.16}$$

Equations (2.15) yield the fully developed velocity and thermal profiles neglecting the viscous and Joulean dissipation.* The inclusion of equations (2.16) refines this approximation to include the dissipation functions.

* Note that the case $\lambda = -1$ is that treated by Gershuni and Zhukhovitskii [3].

For the zeroth-order functions U_0 and Θ_0 we obtain

$$\Theta_0 = 1 + b_1 Y + b_2 Y^2, \tag{2.17}$$

$$M^3 U_0 = B_0 S_0(M, Y) + B_1 S_1(M, Y) + b_1 + b_2 Y^2, \tag{2.18}$$

where

$$S_0(M, Y) = \sinh M - \sinh MY + \sinh M(Y-1).$$

$$S_1(M, Y) = \sinh M + \sinh MY + \sinh M(Y-1),$$

$$b_1 = \tfrac{1}{2}(a + 2\lambda - 2), \quad b_2 = -a/2,$$

$$B_2 = \frac{M^2(6 + 6\lambda + a - 12a/M^2)}{24(\cosh M - 1)}$$

and $$B_2 = \frac{-M(\lambda - 1)}{\sinh M}.$$

The first-order functions are obviously more tedious to evaluate. They can be expressed as

$$M^4 \Theta_1 = -\sum_{n=1}^{\infty} C_n g_n(Y),$$

$$M^4 U_1 = \sum_{n=1}^{\infty} C_n f_n(Y) \tag{2.19}$$

where

$$C_1 = B_1^2, C_2 = B_2^2, C_3 = 2B_1B_2, C_4 = 2B_1, C_5 = 2B_2 \text{ and } C_6 = 1.$$

The subsidiary functions $g_n(Y)$ and $f_n(Y)$ are:

$$4M^2 g_1 = \cosh 2M(Y-1) - 2\cosh 2M(Y - \tfrac{1}{2}) + \cosh 2MY - (\cosh 2M - 2\cosh M + 1), \tag{2.20}$$

$$12M^4 f_1 = 4M^2 g_1 + \left\{\sinh M + \frac{3M(\cosh 2M - 2\cosh M + 1)}{2(\cosh M - 1)}\right\} S_0(M, Y), \tag{2.21}$$

$$4M^2 g_2 = \cosh 2M(Y-1) + 2\cosh 2M(Y - \tfrac{1}{2}) + \cosh 2MY - (\cosh 2M + 2\cosh M + 1), \tag{2.22}$$

$$12M^4 f_2 = 4M^2 g_2 + \left\{\sinh M \frac{(\cosh M + 1)}{(\cosh M - 1)} + \frac{3M(\cosh 2M + 2\cosh M + 1)}{2(\cosh M - 1)}\right\} S_0(M, Y), \tag{2.23}$$

$$4M^2 g_3 = (1 - 2Y)(1 - \cosh 2M) + \cosh 2M(Y-1) - \cosh 2MY, \tag{2.24}$$

$$12M^4 f_3 = 4M^2 g_3 - 4(1 - \cosh 2M)\left\{\frac{(\cosh M + 1)}{\sinh M} \sinh MY - \cosh MY + 1 - 2Y\right\} \tag{2.25}$$

$$M^2 g_4 = M\left[\left(b_0 + \frac{2b_2}{M^2}\right)\sinh M + (b_1 + b_2) Y \sinh M + \left\{b_0 + \frac{2b_2}{M^2} + b_1 Y + b_2 Y^2\right\} S_2(M, Y)\right] + b_1(\cosh M - 1) - 2(b_1 - b_2)(\cosh M - 1)Y - (b_1 + 2b_2 Y)C_0(M, Y), \tag{2.26}$$

$$M^2 g_5 = M\left[\left(b_0 + \frac{2b_2}{M^2}\right)\sinh M + Y\left(b_1 + b_2 - \frac{8b_2}{M^2}\right)\sinh M + \left\{b_0 + \frac{2b_2}{M^2} + b_1 Y + b_2 Y^2\right\} S_3(M, Y)\right] + b_1(\cosh M + 1) + 2b_2(\cosh M + 1)Y - (b_1 + 2b_2 Y)C_1(M, Y). \tag{2.27}$$

$$
\left.
\begin{aligned}
12Mf_p &= \frac{12M}{\sinh M}\left[\frac{2-2\cosh M+M\sinh M}{2(1-\cosh M)M^2}\left(b_0+\frac{b_1}{2}\right)-\frac{b_0}{M^2}\right]S_0(M,\,Y)\\
&+\frac{12b_1}{M}\left[\frac{\sinh MY}{\sinh M}-Y\right]+\left[\frac{3\delta(b_3+b_4)}{2M}-\left(3b_5+\tfrac{3}{2}b_3+b_4+\frac{3b_4}{2M^2}\right)\left(\frac{1+\delta\cosh M}{\sinh M}\right)\right.\\
&+\left\{\delta\left(6b_5+2b_4+3b_3+\frac{1}{M^2}(9b_3+21b_4)\right)+\frac{9}{M^2}b_3\right\}\frac{\sinh M}{2(\cosh M-1)}\\
&\left.+\left(6b_5+\frac{21b_4}{M^2}\right)\frac{\cosh M+\delta}{2M(\cosh M-1)}-\left(6b_5+9b_4+9b_3+\frac{21b_4}{M^2}\right)\frac{1+\delta\cosh M}{2M(\cosh M-1)}\right]\\
&\times\frac{S_0(M,\,Y)}{\sinh M}+\left[\frac{3\delta}{M}(b_3+b_4)+\left(6b_5+3b_3+2b_4+\frac{3b_4}{M^2}\right)\frac{1+\delta\cosh M}{\sinh M}\right]\sinh MY\\
&+\left[\left(6b_5+\frac{3b_4}{M^2}\right)Y+3b_3Y^2+2b_4Y^3\right][\cosh M(Y-1)+\delta\cosh MY]-\\
&-\frac{3}{M}[b_3Y+b_4Y^2][\sinh M(Y-1)+\delta\sinh MY]+3\left\{\frac{b_7}{2M}\,\frac{1+\delta\cosh M}{\sinh M}\right.\\
&-\frac{\delta}{2}(2b_6+b_7)-[\delta(2b_6+3b_7)-2b_6]\frac{\sinh M}{2M(\cosh M-1)}\\
&\left.-\left(1+\frac{2}{M^2}\right)b_7\frac{\cosh M+\delta}{2M^2(\cosh M-1)}+\left(2b_6+b_7+\frac{b_7}{M}+\frac{2b_7}{M^2}\right)\frac{1+\delta\cosh M}{2(\cosh M-1)}\right\}S_0(M,\,Y)\\
&+3\left[b_7\frac{(1+\delta\cosh M)}{\sinh M}-\delta(2b_6+b_7)\right]\sinh MY+3[2b_6Y+b_7Y^2]\,[\sinh M(Y-1)\\
&+\delta\sinh MY]-\frac{3b_7}{M}Y[\cosh M(Y-1)+\delta\cosh MY],
\end{aligned}
\right\}\quad(2.28)
$$

$$
g_6=\sum_{r=0}^{4}d_rY^{r+2}-Y\sum_{r=0}^{4}d_r, \tag{2.29}
$$

$$
\left.
\begin{aligned}
f_6 &= \left[-\frac{1}{M^2\sinh M}\left\{(3d_1+6d_2+10d_3+15d_4+)\,\frac{1}{M^2}\left(60d_3+180d_4\right)\right\}\right.\\
&+\frac{M}{2(\cosh M-1)}\left\{(-\tfrac{1}{6}d_0+\tfrac{1}{4}d_1+\tfrac{3}{10}d_2+\tfrac{1}{3}d_3+\tfrac{5}{14}d_4)\right.\\
&+\frac{1}{M^2}(2d_0+3d_1+4d_2+5d_3+6d_4)+\frac{1}{M^4}(24d_2+60d_3+120d_4)\\
&\left.\left.+\frac{720d_4}{M^6}\right\}\right]S_0(M,\,Y)+\frac{1}{M^2\sinh M}\left[(6d_1+12d_2+20d_3+30d_4)\right.\\
&\left.+\frac{1}{M^2}(120d_3+360d_4)\right]\sinh MY+Y\left(\sum_{r=0}^{4}d_r-\frac{6d_1}{M^2}-\frac{120d_3}{M^4}\right)\\
&-Y^2\left(d_0+\frac{12d_2}{M^2}+\frac{360d_4}{M^4}\right)-Y^3\left(d_1+\frac{20d_3}{M^2}\right)-d_3Y^4-d_4Y^5.
\end{aligned}
\right\}\quad(2.30)
$$

Here $S_2(M, Y) = \sinh M(Y - 1) - \sinh MY$, $S_3(M, Y) = \sinh M(Y - 1) + \sinh MY$, $C_0(M, Y) = \cosh M(Y - 1) - \cosh MY$, $C_1(M, Y) = \cosh M(Y - 1) + \cosh MY$; in equations (2.27) $\delta = -1$ when $p = 4$ and $\delta = +1$ when $p = 5$, and the constants b_r are

$$b_0 = 1 - \frac{\alpha}{M^2},\ b_3 = \frac{b_1}{M},\ b_4 = \frac{b_2}{M^2},$$

$$b_5 = \frac{1}{M}\left(b_0 + \frac{2b_2}{M^2}\right),\ b_6 = -\frac{b_1}{M^2}$$

and
$$b_7 = -\frac{2b_2}{M^2}.$$

Finally in equations (2.28) and (2.29) the constants d_r are

$$d_0 = \tfrac{1}{2}(b_1^2 + M^2 b_0),\ d_1 = \tfrac{1}{3} b_1(b_2 + M^2 b_0),$$

$$d_2 = \tfrac{1}{12}\{4b_2^2 + M^2(2b_0b_2 + b_1^2)\},$$

$$d_3 = \tfrac{1}{10} M^2 b_1 b_2 \text{ and } d_4 = \tfrac{1}{30} M^2 b_2^2.$$

The zeroth and first-order functions have been calculated from equations (2.17) to (2.30) for the following cases: $M = 0, 2, 4$ and 10, $\alpha = 0, 10$, and 100 and $\lambda = -1(1)2$. These are tabulated at an interval of $Y = 0{\cdot}2$ in Tables 2 and 3. Note that in these tables and in all succeeding tables the figure in parenthesis, say n, denotes a multiplying factor of 10^{-n}.

2(c) *Flow and heat transfer characteristics*

From section 2(b) quantitative information may now be deduced for flow and heat transfer characteristics such as the net mass flow, the induced magnetic flux density, and the heat transfer coefficients at the wall.

(i) *Net mass flow.* The net mass flow per sec per unit breadth of wall in the z-direction when the isothermal walls are a distance a cm apart is given by

$$\bar{q} = \mu \bar{U}/PK_A \qquad (2.31a)$$

where on using (2.14) we obtain

$$\bar{U} = K\int_0^1 U_0 \mathrm{d}Y + K^2 \int_0^1 U_1 \mathrm{d}Y + 0(K^3). \qquad (2.31b)$$

These integrals have already been evaluated in the determination of U_0 and U_1 and are tabulated in Table 4 for $\alpha = 0$, 10 and 100, $\lambda = -1(1)2$ and $M = 0, 2, 4, 10, 20, 40, 100$ and 200.

(ii) *Induced magnetic flux density.* From Ohm's law as stated in equation (2.3) together with expressions (2.7), (2.10) and (2.14) the induced magnetic flux density $B_x = \mu_e H_x$ is evaluated to be

$$B_x = B_0 G_M \left[\int_0^Y (\bar{U}_0 - U_0)\mathrm{d}Y + K\int_0^Y (\bar{U}_1 - U_1)\mathrm{d}Y + 0(K^2)\right]. \qquad (2.32)$$

Here $G_M = 4\pi\sigma\mu_e\nu G$ is the magnetic Grashof number and is the natural convection flow equivalent of the magnetic Reynolds number. The magnitude of B_x can then be calculated using equation (2.31) and Tables 2–4.

(iii) *Heat transfer coefficients.* The heat transfer coefficients at the wall can be expressed in the usual fashion by dimensionless Nusselt numbers. For the case when the walls are not at the same temperature i.e. $\lambda = 1$ then at the wall $y = 0$

$$Nu_0 = \left(\frac{\mathrm{d}T}{\mathrm{d}y}\right)_{y=0} \frac{a}{(T_{w1} - T_{w0})} = \frac{1}{(\lambda - 1)K}\left(\frac{\partial \Theta}{\partial Y}\right)_{Y=0}, \qquad (2.33)$$

and at the wall $y = a$

$$Nu_1 = \left(\frac{\mathrm{d}T}{\mathrm{d}y}\right)_{y=a} \frac{a}{(T_{w1} - T_{w0})} = \frac{1}{(\lambda - 1)K}\left(\frac{\partial \Theta}{\partial Y}\right)_{Y=1}. \qquad (2.34)$$

When the series expansion for Θ given by equations (2.14), (2.17) and (2.19) is substituted

Table 2

	u_0, $\alpha = 0$, $\lambda = 2$				u_1, $\alpha = 0$, $\lambda = 2$			
y	$M = 0$	$M = 2$	$M = 4$	$M = 10$	$M = 0$	$M = 2$	$M = 4$	$M = 10$
0·0	0	0	0	0	0	0	0	0
0·2	1·12 (−1)	1·07 (−1)	9·51 (−2)	6·25 (−2)	8·47 (−4)	7·23 (−4)	4·89 (−4)	1·25 (−4)
0·4	1·76 (−1)	1·63 (−1)	1·36 (−1)	7·25 (−2)	1·32 (−3)	1·10 (−3)	7·01 (−4)	1·44 (−4)
0·6	1·84 (−1)	1·70 (−1)	1·41 (−1)	7·44 (−2)	1·32 (−3)	1·10 (−3)	7·03 (−4)	1·44 (−4)
0·8	1·28 (−1)	1·21 (−1)	1·07 (−1)	6·71 (−2)	8·57 (−4)	7·31 (−4)	4·95 (−4)	1·23 (−4)
1·0	0	0	0	0	0	0	0	0

	u_0, $\alpha = 0$, $\lambda = 1$				u_1, $\alpha = 0$, $\lambda = 1$			
y	$M = 0$	$M = 2$	$M = 4$	$M = 10$	$M = 0$	$M = 2$	$M = 4$	$M = 10$
0·0	0	0	0	0	0	0	0	0
0·2	8·00 (−2)	7·61 (−2)	6·73 (−2)	4·32 (−2)	3·75 (−4)	3·20 (−4)	2·17 (−4)	5·4 (−5)
0·4	1·20 (−1)	1·11 (−1)	9·24 (−2)	4·90 (−2)	5·82 (−4)	4·85 (−4)	3·09 (−4)	6·3 (−5)
0·6	1·20 (−1)	1·11 (−1)	9·24 (−2)	4·90 (−2)	5·82 (−4)	4·85 (−4)	3·09 (−4)	6·3 (−5)
0·8	8·00 (−2)	7·61 (−2)	6·73 (−2)	4·32 (−2)	3·75 (−4)	3·20 (−4)	2·17 (−4)	5·4 (−5)
1·0	0	0	0	0	0	0	0	0

	u_0, $\alpha = 0$, $\lambda = 0$				u_1, $\alpha = 0$, $\lambda = 0$			
y	$M = 0$	$M = 2$	$M = 4$	$M = 10$	$M = 0$	$M = 2$	$M = 4$	$M = 10$
0·0	0	0	0	0	0	0	0	0
0·2	4·80 (−2)	4·53 (−2)	3·94 (−2)	2·39 (−2)	1·03 (−4)	8·8 (−5)	6·0 (−5)	2·1 (−5)
0·4	6·40 (−2)	5·92 (−2)	4·89 (−2)	2·54 (−2)	1·58 (−4)	1·32 (−4)	8·5 (−5)	1·7 (−5)
0·6	5·60 (−2)	5·20 (−2)	4·35 (−2)	2·36 (−2)	1·57 (−4)	1·31 (−4)	8·4 (−5)	1·8 (−5)
0·8	3·20 (−2)	3·08 (−2)	2·79 (−2)	1·93 (−2)	1·00 (−4)	8·5 (−5)	5·8 (−5)	1·5 (−5)
1·0	0	0	0	0	0	0	0	0

	u_0, $\alpha = 0$, $\lambda = -1$				u_1, $\alpha = 0$, $\lambda = -1$			
y	$M = 0$	$M = 2$	$M = 4$	$M = 10$	$M = 0$	$M = 2$	$M = 4$	$M = 10$
0·0	0	0	0	0	0	0	0	0
0·2	1·60 (−2)	1·46 (−2)	1·15 (−2)	4·65 (−3)	3·1 (−5)	2·6 (−5)	1·9 (−5)	9·0 (−6)
0·4	8·0 (−3)	7·17 (−3)	5·42 (−3)	1·84 (−3)	4·9 (−5)	4·2 (−5)	2·8 (−5)	1·1 (−5)
0·6	−8·0 (−3)	−7·17 (−3)	−5·42 (−3)	−1·84 (−3)	4·9 (−5)	4·2 (−5)	2·8 (−5)	1·1 (−5)
0·8	−1·60 (−2)	−1·46 (−2)	−1·15 (−2)	−4·65 (−3)	3·1 (−5)	2·6 (−5)	1·9 (−5)	9·0 (−6)
1·0	0	0	0	0	0	0	0	0

Table 2 (contd.)

	u_0, $\alpha = 10$, $\lambda = 2$				u_1, $\alpha = 10$, $\lambda = 2$			
y	$M = 0$	$M = 2$	$M = 4$	$M = 10$	$M = 0$	$M = 2$	$M = 4$	$M = 10$
0·0	0	0	0	0	0	0	0	0
0·2	1·89 (−1)	1·80 (−1)	1·59 (−1)	1·02 (−1)	2·49 (−3)	2·12 (−3)	1·43 (−3)	3·51 (−4)
0·4	3·00 (−1)	2·78 (−1)	2·30 (−1)	1·20 (−1)	3·88 (−3)	3·23 (−3)	2·05 (−3)	4·18 (−4)
0·6	3·08 (−1)	2·85 (−1)	2·36 (−1)	1·22 (−1)	3·88 (−3)	3·23 (−3)	2·05 (−3)	4·11 (−4)
0·8	2·05 (−1)	1·95 (−1)	1·71 (−1)	1·07 (−1)	2·51 (−3)	2·13 (−3)	1·44 (−3)	3·50 (−4)
1·0	0	0	0	0	0	0	0	0

	u_0, $\alpha = 10$, $\lambda = 1$				u_1, $\alpha = 10$, $\lambda = 1$			
y	$M = 0$	$M = 2$	$M = 4$	$M = 10$	$M = 0$	$M = 2$	$M = 4$	$M = 10$
0·0	0	0	0	0	0	0	0	0
0·2	1·57 (−1)	1·49 (−1)	1·31 (−1)	8·29 (−2)	1·62 (−3)	1·38 (−3)	9·26 (−4)	2·21 (−4)
0·4	2·44 (−1)	2·26 (−1)	1·87 (−1)	9·69 (−2)	2·52 (−3)	2·09 (−3)	1·32 (−3)	2·64 (−4)
0·6	2·44 (−1)	2·26 (−1)	1·87 (−1)	9·69 (−2)	2·52 (−3)	2·09 (−3)	1·32 (−3)	2·64 (−4)
0·8	1·57 (−1)	1·49 (−1)	1·31 (−1)	8·29 (−2)	1·62 (−3)	1·38 (−3)	9·26 (−4)	2·22 (−4)
1·0	0	0	0	0	0	0	0	0

	u_0, $\alpha = 10$, $\lambda = 0$				u_1, $\alpha = 10$, $\lambda = 0$			
y	$M = 0$	$M = 2$	$M = 4$	$M = 10$	$M = 0$	$M = 2$	$M = 4$	$M = 10$
0·0	0	0	0	0	0	0	0	0
0·2	1·25 (−1)	1·19 (−1)	1·03 (−1)	6·36 (−2)	9·46 (−4)	8·05 (−4)	5·40 (−4)	1·30 (−4)
0·4	1·88 (−1)	1·74 (−1)	1·43 (−1)	7·33 (−2)	1·47 (−3)	1·22 (−3)	7·71 (−4)	1·56 (−4)
0·6	1·80 (−1)	1·67 (−1)	1·38 (−1)	7·15 (−2)	1·46 (−3)	1·22 (−3)	7·70 (−4)	1·53 (−4)
0·8	1·09 (−1)	1·04 (−1)	9·19 (−2)	5·90 (−2)	9·38 (−4)	7·98 (−4)	5·36 (−4)	1·29 (−4)
1·0	0	0	0	0	0	0	0	0

	u_0, $\alpha = 10$, $\lambda = -1$				u_1, $\alpha = 10$, $\lambda = -1$			
y	$M = 0$	$M = 2$	$M = 4$	$M = 10$	$M = 0$	$M = 2$	$M = 4$	$M = 10$
0	0	0	0	0	0	0	0	0
0·2	9·33 (−2)	8·78 (−2)	7·55 (−2)	4·43 (−2)	4·73 (−4)	4·03 (−4)	2·70 (−4)	6·4 (−5)
0·4	1·32 (−1)	1·22 (−1)	9·97 (−2)	4·97 (−2)	7·31 (−4)	6·09 (−4)	3·85 (−4)	7·6 (−5)
0·6	1·16 (−1)	1·07 (−1)	8·88 (−2)	4·61 (−2)	7·29 (−4)	6·06 (−4)	3·83 (−4)	7·5 (−5)
0·8	6·13 (−2)	5·86 (−2)	5·25 (−2)	3·50 (−2)	4·63 (−4)	3·95 (−4)	2·65 (−4)	6·3 (−5)
1·0	0	0	0	0	0	0	0	0

Table 2 (contd.)

	u_0, $a = 100$, $\lambda = 2$				u_1, $a = 100$, $\lambda = 2$			
y	$M = 0$	$M = 2$	$M = 4$	$M = 10$	$M = 0$	$M = 2$	$M = 4$	$M = 10$
0·0	0	0	0	0	0	0	0	0
0·2	8·85 (−1)	8·39 (−1)	7·35 (−1)	4·59 (−1)	5·67 (−2)	4·82 (−2)	3·22 (−2)	7·63 (−3)
0·4	1·42	1·31	1·08	5·52 (−1)	8·82 (−2)	7·32 (−2)	4·62 (−2)	9·02 (−3)
0·6	1·42	1·32	1·08	5·53 (−1)	8·82 (−2)	7·32 (−2)	4·62 (−2)	9·03 (−3)
0·8	9·01 (−1)	8·53 (−1)	7·47 (−1)	4·64 (−1)	5·67 (−2)	4·82 (−2)	3·23 (−2)	7·63 (−3)
1·0	0	0	0	0	0	0	0	0

	u_0, $a = 100$, $\lambda = 1$				u_1, $a = 100$, $\lambda = 1$			
y	$M = 0$	$M = 2$	$M = 4$	$M = 10$	$M = 0$	$M = 2$	$M = 4$	$M = 10$
0·0	0	0	0	0	0	0	0	0
0·2	8·53 (−1)	8·08 (−1)	7·07 (−1)	4·40 (−1)	5·22 (−2)	4·44 (−2)	2·97 (−2)	7·01 (−3)
0·4	1·36	1·26	1·03	5·28 (−1)	8·12 (−2)	6·74 (−2)	4·25 (−2)	8·29 (−3)
0·6	1·36	1·26	1·03	5·28 (−1)	8·12 (−2)	6·74 (−2)	4·25 (−2)	8·29 (−3)
0·8	8·53 (−1)	8·08 (−1)	7·07 (−1)	4·40 (−1)	5·22 (−2)	4·44 (−2)	2·97 (−2)	7·01 (−3)
1·0	0	0	0	0	0	0	0	0

	u_0, $a = 100$, $\lambda = 0$				u_1, $a = 100$, $\lambda = 0$			
y	$M = 0$	$M = 2$	$M = 4$	$M = 10$	$M = 0$	$M = 2$	$M = 4$	$M = 10$
0·0	0	0	0	0	0	0	0	0
0·2	8·21 (−1)	7·77 (−1)	6·79 (−1)	4·21 (−1)	4·79 (−2)	4·07 (−2)	2·72 (−2)	6·41 (−3)
0·4	1·30	1·20	9·91 (−1)	5·04 (−1)	7·45 (−2)	6·19 (−2)	3·90 (−2)	7·59 (−3)
0·6	1·30	1·20	9·86 (−1)	5·03 (−1)	7·45 (−2)	6·19 (−2)	3·90 (−2)	7·59 (−3)
0·8	8·05 (−1)	7·63 (−1)	6·68 (−1)	4·16 (−1)	4·78 (−2)	4·07 (−2)	2·72 (−2)	6·41 (−3)
1·0	0	0	0	0	0	0	0	0

	u_0, $a = 100$, $\lambda = -1$				u_1, $a = 100$, $\lambda = -1$			
y	$M = 0$	$M = 2$	$M = 4$	$M = 10$	$M = 0$	$M = 2$	$M = 4$	$M = 10$
0·0	0	0	0	0	0	0	0	0
0·2	7·89 (−1)	7·47 (−1)	6·52 (−1)	4·02 (−1)	4·38 (−2)	3·72 (−2)	2·49 (−2)	5·85 (−3)
0·4	1·25	1·15	9·48 (−1)	4·81 (−1)	6·81 (−2)	5·66 (−2)	3·57 (−2)	6·92 (−3)
0·6	1·23	1·14	9·37 (−1)	4·77 (−1)	6·81 (−2)	5·65 (−2)	3·56 (−2)	6·92 (−3)
0·8	7·57 (−1)	7·17 (−1)	6·29 (−1)	3·92 (−1)	4·37 (−2)	3·71 (−2)	2·48 (−2)	5·84 (−3)
1·0	0	0	0	0	0	0	0	0

Table 3

	Θ_0, $\alpha = 0$				Θ_0, $\alpha = 10$				Θ_0, $\alpha = 100$			
y	$\lambda = 2$	$\lambda = 1$	$\lambda = 0$	$\lambda = -1$	$\lambda = 2$	$\lambda = 1$	$\lambda = 0$	$\lambda = -1$	$\lambda = 2$	$\lambda = 1$	$\lambda = 0$	$\lambda = -1$
0·0	1·0	1·0	1·0	1·0	1·0	1·0	1·0	1·0	1·0	1·0	1·0	1·0
0·2	1·2	1·0	0·8	0·6	2·0	1·8	1·6	1·4	9·2	9·0	8·8	8·6
0·4	1·4	1·0	0·6	0·2	2·6	2·2	1·8	1·4	13·4	13·0	12·6	12·2
0·6	1·6	1·0	0·4	−0·2	2·8	2·2	1·6	1·0	13·6	13·0	12·4	11·8
0·8	1·8	1·0	0·2	−0·6	2·6	1·8	1·0	0·2	9·8	9·0	8·2	7·4
1·0	2·0	1·0	0	−1·0	2·0	1·0	0	−1·0	2·0	1·0	0	−1·0

	Θ_1, $\alpha = 0$, $\lambda = 2$				Θ_1, $\alpha = 0$, $\lambda = 1$			
y	$M = 0$	$M = 2$	$M = 4$	$M = 10$	$M = 0$	$M = 2$	$M = 4$	$M = 10$
0·0	0	0	0	0	0	0	0	0
0·2	1·01 (−2)	8·97 (−3)	6·78 (−3)	2·57 (−3)	4·53 (−3)	4·04 (−3)	3·05 (−3)	1·16 (−3)
0·4	1·19 (−2)	1·07 (−2)	8·29 (−3)	3·30 (−3)	5·20 (−3)	4·70 (−3)	3·66 (−3)	1·47 (−3)
0·6	1·18 (−2)	1·07 (−2)	8·36 (−3)	3·39 (−3)	5·20 (−3)	4·70 (−3)	3·66 (−3)	1·47 (−3)
0·8	1·05 (−2)	9·36 (−3)	7·07 (−3)	2·71 (−3)	4·53 (−3)	4·04 (−3)	3·05 (−3)	1·16 (−3)
1·0	0	0	0	0	0	0	0	0

	Θ_1, $\alpha = 0$, $\lambda = 0$				Θ_1, $\alpha = 0$, $\lambda = -1$			
y	$M = 0$	$M = 2$	$M = 4$	$M = 10$	$M = 0$	$M = 2$	$M = 4$	$M = 10$
0·0	0	0	0	0	0	0	0	0
0·2	1·28 (−3)	1·14 (−3)	8·71 (−4)	3·42 (−4)	2·97 (−4)	2·80 (−4)	2·38 (−4)	1·13 (−4)
0·4	1·41 (−3)	1·28 (−3)	1·02 (−3)	4·20 (−4)	4·87 (−4)	4·46 (−4)	3·55 (−4)	1·46 (−4)
0·6	1·43 (−3)	1·29 (−3)	9·92 (−4)	3·88 (−4)	4·87 (−4)	4·46 (−4)	3·55 (−4)	1·46 (−4)
0·8	1·13 (−3)	1·01 (−3)	7·74 (−4)	2·96 (−4)	2·97 (−4)	2·80 (−4)	2·38 (−4)	1·13 (−4)
1·0	0	0	0	0	0	0	0	0

	Θ_1, $\alpha = 10$, $\lambda = 2$				Θ_1, $\alpha = 10$, $\lambda = 1$			
y	$M = 0$	$M = 2$	$M = 4$	$M = 10$	$M = 0$	$M = 2$	$M = 4$	$M = 10$
0·0	0	0	0	0	0	0	0	0
0·2	2·96 (−2)	2·64 (−2)	1·98 (−2)	7·34 (−3)	1·94 (−2)	1·73 (−2)	1·29 (−2)	4·77 (−3)
0·4	3·49 (−2)	3·15 (−2)	2·44 (−2)	9·52 (−3)	2·26 (−2)	2·04 (−2)	1·58 (−2)	6·19 (−3)
0·6	3·48 (−2)	3·15 (−2)	2·45 (−2)	9·67 (−3)	2·26 (−2)	2·04 (−2)	1·58 (−2)	6·19 (−3)
0·8	3·03 (−2)	2·70 (−2)	2·03 (−2)	7·54 (−3)	1·94 (−2)	1·73 (−2)	1·29 (−2)	4·77 (−3)
1·0	0	0	0	0	0	0	0	0

Table 3 (*contd.*)

	Θ_1, $a = 10$, $\lambda = 0$				Θ_1, $a = 10$, $\lambda = 1$			
y	$M = 0$	$M = 2$	$M = 4$	$M = 10$	$M = 0$	$M = 2$	$M = 4$	$M = 10$
0·0	0	0	0	0	0	0	0	0
0·2	1·14 (−2)	1·02 (−2)	7·62 (−3)	2·80 (−3)	5·75 (−3)	5·11 (−3)	3·83 (−3)	1·41 (−3)
0·4	1·31 (−2)	1·19 (−2)	9·23 (−3)	3·62 (−3)	6·55 (−3)	5·94 (−3)	4·65 (−3)	1·84 (−3)
0·6	1·33 (−2)	1·19 (−2)	9·19 (−3)	3·54 (−3)	6·73 (−3)	6·04 (−3)	4·61 (−3)	1·73 (−3)
0·8	1·11 (−2)	9·83 (−3)	7·37 (−3)	2·69 (−3)	5·26 (−3)	4·69 (−3)	3·54 (−3)	1·29 (−3)
1·0	0	0	0	0	0	0	0	0

	Θ_1, $a = 100$, $\lambda = 2$				Θ_1, $a = 100$, $\lambda = 1$			
y	$M = 0$	$M = 2$	$M = 4$	$M = 10$	$M = 0$	$M = 2$	$M = 4$	$M = 10$
0·0	0	0	0	0	0	0	0	0
0·2	6·74 (−1)	5·99 (−1)	4·47 (−1)	1·61 (−1)	6·22 (−1)	5·52 (−1)	4·12 (−1)	1·48 (−1)
0·4	7·96 (−1)	7·18 (−1)	5·54 (−1)	2·13 (−1)	7·33 (−1)	6·61 (−1)	5·10 (−1)	1·96 (−1)
0·6	7·95 (−1)	7·17 (−1)	5·54 (−1)	2·14 (−1)	7·33 (−1)	6·61 (−1)	5·10 (−1)	1·96 (−1)
0·8	6·77 (−1)	6·01 (−1)	4·49 (−1)	1·62 (−1)	6·22 (−1)	5·52 (−1)	4·12 (−1)	1·48 (−1)
1·0	0	0	0	0	0	0	0	0

	Θ_1, $a = 100$, $\lambda = 0$				Θ_1, $a = 100$, $\lambda = -1$			
y	$M = 0$	$M = 2$	$M = 4$	$M = 10$	$M = 0$	$M = 2$	$M = 4$	$M = 10$
0·0	0	0	0	0	0	0	0	0
0·2	5·71 (−1)	5·07 (−1)	3·78 (−1)	1·36 (−1)	5·23 (−1)	4·64 (−1)	3·46 (−1)	1·24 (−1)
0·4	6·72 (−1)	6·06 (−1)	4·68 (−1)	1·80 (−1)	6·14 (−1)	5·54 (−1)	4·28 (−1)	1·65 (−1)
0·6	6·73 (−1)	6·07 (−1)	4·68 (−1)	1·79 (−1)	6·16 (−1)	5·55 (−1)	4·28 (−1)	1·64 (−1)
0·8	5·69 (−1)	5·05 (−1)	3·77 (−1)	1·35 (−1)	5·18 (−1)	4·60 (−1)	3·43 (−1)	1·14 (−1)
1·0	0	0	0	0	0	0	0	0

Table 4

	$\int_0^1 u_0\,dY$, $\alpha = 0$				$\int_0^1 u_1\,dY$, $\alpha = 0$			
M	$\lambda = 2$	$\lambda = 1$	$\lambda = 0$	$\lambda = -1$	$\lambda = 2$	$\lambda = 1$	$\lambda = 0$	$\lambda = -1$
0	1·25 (−1)	8·33 (−2)	4·17 (−2)	0	9·01 (−4)	3·97 (−4)	1·07 (−4)	3·31 (−5)
2	1·17 (−1)	7·83 (−2)	3·91 (−2)	0	7·60 (−4)	3·35 (−4)	9·1 (−5)	2·9 (−5)
4	1·01 (−1)	6·72 (−2)	3·36 (−2)	0	5·00 (−4)	2·20 (−4)	6·0 (−5)	2·0 (−5)
10	6·00 (−2)	4·00 (−2)	2·00 (−2)	0	1·15 (−4)	5·1 (−5)	1·4 (−5)	5·0 (−6)
20	3·37 (−2)	2·25 (−2)	1·12 (−2)	0	2·1 (−5)	9·0 (−6)	3·0 (−6)	1·0 (−6)
40	1·78 (−2)	1·19 (−2)	5·94 (−3)	0	3·0 (−6)	1·0 (−6)	0	0
100	7·35 (−3)	4·90 (−3)	2·45 (−3)	0	—	—	—	—
200	3·71 (−3)	2·47 (−3)	1·24 (−3)	0	—	—	—	—

	$\int_0^1 u_0\,dY$, $\alpha = 10$				$\int_0^1 u_1\,dY$, $\alpha = 10$			
M	$\lambda = 2$	$\lambda = 1$	$\lambda = 0$	$\lambda = -1$	$\lambda = 2$	$\lambda = 1$	$\lambda = 0$	$\lambda = -1$
0	2·08 (−1)	1·67 (−1)	1·25 (−1)	8·33 (−2)	2·65 (−3)	1·71 (−3)	9·98 (−4)	4·97 (−4)
2	1·95 (−1)	1·56 (−1)	1·17 (−1)	7·79 (−2)	2·23 (−3)	1·44 (−3)	8·40 (−4)	4·18 (−4)
4	1·67 (−1)	1·33 (−1)	9·97 (−2)	6·61 (−2)	1·46 (−3)	9·42 (−4)	5·47 (−4)	2·73 (−4)
10	9·77 (−2)	7·77 (−2)	5·77 (−2)	3·77 (−2)	3·27 (−4)	2·10 (−4)	1·21 (−4)	6·0 (−5)
20	5·40 (−2)	4·28 (−2)	3·15 (−2)	2·03 (−2)	5·7 (−5)	3·6 (−5)	2·1 (−5)	1·0 (−5)
40	2·82 (−2)	2·22 (−2)	1·63 (−2)	1·03 (−2)	8·0 (−6)	5·0 (−6)	3·0 (−6)	1·0 (−6)
100	1·15 (−2)	9·06 (−3)	6·61 (−3)	4·16 (−3)	—	—	—	—
200	5·79 (−3)	4·56 (−3)	3·32 (−3)	2·08 (−3)	—	—	—	—

	$\int_0^1 u_0\,dY$, $\alpha = 100$				$\int_0^1 u_1\,dY$, $\alpha = 100$			
M	$\lambda = 2$	$\lambda = 1$	$\lambda = 0$	$\lambda = -1$	$\lambda = 2$	$\lambda = 1$	$\lambda = 0$	$\lambda = -1$
0	9·58 (−1)	9·17 (−1)	8·75 (−1)	8·33 (−1)	6·01 (−2)	5·53 (−2)	5·07 (−2)	4·64 (−2)
2	8·96 (−1)	8·57 (−1)	8·18 (−1)	7·79 (−1)	5·05 (−2)	4·65 (−2)	4·27 (−2)	3·90 (−2)
4	7·62 (−1)	7·28 (−1)	6·94 (−1)	6·61 (−1)	3·28 (−2)	3·02 (−2)	2·77 (−2)	2·53 (−2)
10	4·37 (−1)	4·17 (−1)	3·97 (−1)	3·77 (−1)	7·18 (−3)	6·60 (−3)	6·04 (−3)	5·50 (−3)
20	2·36 (−1)	2·25 (−1)	2·14 (−1)	2·03 (−1)	1·15 (−3)	1·05 (−3)	9·57 (−4)	8·68 (−4)
40	1·21 (−1)	1·15 (−1)	1·09 (−1)	1·03 (−1)	1·71 (−4)	1·57 (−4)	1·42 (−4)	1·29 (−4)
100	4·90 (−2)	4·65 (−2)	4·41 (−2)	4·16 (−2)	—	—	—	—
200	2·45 (−2)	2·33 (−2)	2·21 (−2)	2·08 (−2)	—	—	—	—

into equations (2.33) and (2.34) the following expressions are obtained:

$$
\left.\begin{aligned}
Nu_0 = \frac{(a + 2\lambda - 2)}{2(\lambda - 1)} + \frac{K}{(\lambda - 1)} \Big[D_0(a, \lambda, M) + (\lambda - 1)\left(\lambda + 3 - \frac{8a}{M^2}\right)\frac{\cosh M + 1}{4M^3 \cosh M} \\
- D_1(a, \lambda, M) + \frac{1}{240M^2}(a^2 + 8a\lambda + 20\lambda^2 - 8a - 80\lambda - 60) \\
+ \frac{1}{24M^2}(a^2 + 8a\lambda - 12\lambda^2 + 16a - 24\lambda + 36) - \frac{1}{2M^6}(a^2 - 8a\lambda + 8a)\Big],
\end{aligned}\right\} \quad (2.35)
$$

and

$$
\left.\begin{aligned}
Nu_1 = \frac{(2\lambda - a - 2)}{2(\lambda - 1)} + \frac{K}{\lambda - 1} \Big[D_0(a, \lambda, M) + (\lambda - 1)\left(3\lambda + 1 - \frac{8a}{M^2}\right)\frac{\cosh M + 1}{4M^3 \cosh M} \\
- D_1(a, \lambda, M) - \frac{1}{240M^2}(a^2 - 8a\lambda - 60\lambda^2 + 8a - 80\lambda + 20) \\
- \frac{1}{24M^2}(a^2 + 8a\lambda - 12\lambda^2 + 16a - 24\lambda + 36) - \frac{1}{2M^6}(a^2 - 8a\lambda + 8a)\Big],
\end{aligned}\right\} \quad (2.36)
$$

where

$$D_0(a, \lambda, M) = \left(6 + 6\lambda + a - \frac{12a}{M^2}\right)^2 \frac{\sinh M}{576M(\cosh M - 1)},$$

and

$$D_1(a, \lambda, M) = (\lambda - 1)\left(6 + 6\lambda + a - \frac{12a}{M^2}\right)\left\{\frac{(M^2 - 4)\sinh 2M + M(\cosh 2M - 1) + 8\sinh M}{48M^4 \sinh M(\cosh M - 1)}\right\}.$$

When the walls are at equal temperature i.e. $\lambda = 1$ the Nusselt number may be defined as

$$
\left.\begin{aligned}
Nu = Nu_0 = Nu_1 &= \frac{a}{(T_{w0} - T_s)}\left(\frac{dT}{dy}\right)_0 \\
&= \frac{1}{K}\left(\frac{\partial \Theta}{\partial Y}\right)_{Y=0}
\end{aligned}\right\} \quad (2.37)
$$

On using the results of section 2(b) we obtain when $\lambda = 1$,

$$
\left.\begin{aligned}
Nu = \frac{a}{2} + K\Big[D_0(a, 1, M) \\
+ \frac{1}{240M^2}(a^2 - 120) \\
+ \frac{1}{24M^4}a(a + 24) - \frac{a^2}{2M^6}\Big].
\end{aligned}\right\} \quad (2.38)
$$

The Nusselt number $Nu_{0,1}^{(0)}$ due to ordinary conduction and $Nu_{0,1}^{(1)}$ due to dissipation effects have been evaluated for $a = 0$, 10 and 100, $\lambda = -1(1)2$ and $M = 0, 2, 4, 10, 50(50)200$ and are given in Table 5. The actual wall Nusselt number is then given by

$$Nu_{0,1} = Nu_{0,1}^{(0)} + K \quad Nu_{0,1}^{(1)} + 0(K^2). \quad (2.39)$$

2(d) *Discussion*

The velocity and thermal profiles for fully developed flow depend on the four parameters K, a, λ and M. Let us first consider M fixed and examine general trends due to variations in K, a and λ. These are the same as in the magnetic field-free case and have been summarized by Ostrach [1] as follows:

(i) An increase in either the wall temperature difference ratio λ or the heat source parameter a increases the velocity, net mass flow and temperature.

(ii) The viscous dissipation was found to alter appreciably the velocity and temperature profiles in some cases, showing that as K was increased (a and λ remaining fixed) then the velocity was

Table 5

	$Nu_0^{(0)}$			$Nu_1^{(0)}$		
λ	$a = 0$	$a = 10$	$a = 100$	$a = 0$	$a = 10$	$a = 100$
2	1	6	51	1	4	49
1	0	5	50	0	5	50
0	1	4	49	1	6	51
−1	1	1·5	24	1	3·5	26

	$Nu_0^{(1)}$, $a = 0$				$Nu_1^{(1)}$, $a = 0$			
M	$\lambda = 2$	$\lambda = 1$	$\lambda = 0$	$\lambda = -1$	$\lambda = 2$	$\lambda = 1$	$\lambda = 0$	$\lambda = -1$
0	8·61 (−2)	4·16 (−2)	−1·39 (−2)	−1·39 (−3)	−1·02 (−1)	4·16 (−2)	8·33 (−3)	1·39 (−3)
2	8·14 (−2)	3·91 (−2)	−1·28 (−2)	−1·27 (−3)	−9·59 (−2)	3·91 (−2)	7·99 (−3)	1·27 (−3)
4	7·08 (−2)	3·35 (−2)	−1·07 (−2)	−1·01 (−3)	−8·13 (−2)	3·35 (−2)	7·15 (−3)	1·01 (−3)
10	4·35 (−2)	2·00 (−2)	−5·80 (−3)	−4·35 (−4)	−4·70 (−2)	2·00 (−2)	4·63 (−3)	4·35 (−4)
50	1·07 (−2)	4·80 (−3)	−1·21 (−3)	−2·9 (−4)	−1·09 (−2)	4·80 (−3)	1·18 (−3)	2·9 (−4)
100	5·49 (−3)	2·45 (−3)	−6·24 (−4)	−8·0 (−5)	−5·54 (−3)	2·45 (−3)	6·08 (−4)	8·0 (−5)
200	2·78 (−3)	1·23 (−3)	−3·12 (−4)	−2·0 (−6)	−2·43 (−3)	1·23 (−3)	3·27 (−4)	2·0 (−6)

	$Nu_0^{(1)}$, $a = 10$				$Nu_1^{(1)}$, $a = 10$			
M	$\lambda = 2$	$\lambda = 1$	$\lambda = 0$	$\lambda = -1$	$\lambda = 2$	$\lambda = 1$	$\lambda = 0$	$\lambda = -1$
0	2·48 (−1)	1·67 (−1)	1·03 (−1)	2·74 (−2)	2·75 (−1)	1·67 (−1)	8·72 (−2)	1·75 (−2)
2	2·33 (−1)	1·56 (−1)	9·53 (−2)	2·52 (−2)	2·56 (−1)	1·56 (−1)	8·19 (−2)	1·66 (−2)
4	2·00 (−1)	1·32 (−1)	7·96 (−2)	2·04 (−2)	2·16 (−1)	1·32 (−1)	7·01 (−2)	1·44 (−2)
10	1·17 (−1)	7·56 (−2)	4·27 (−2)	2·53 (−3)	1·22 (−1)	7·56 (−2)	4·06 (−2)	8·54 (−3)
50	2·67 (−2)	1·67 (−3)	9·17 (−3)	1·87 (−3)	2·70 (−2)	1·67 (−3)	8·90 (−3)	2·13 (−3)
100	1·34 (−2)	8·39 (−3)	4·49 (−3)	9·03 (−4)	1·36 (−2)	8·39 (−3)	4·45 (−3)	8·86 (−4)
150	9·03 (−3)	5·60 (−3)	2·98 (−3)	5·95 (−4)	9·06 (−3)	5·60 (−3)	2·97 (−3)	5·88 (−4)
200	6·78 (−3)	4·19 (−3)	2·23 (−3)	4·43 (−4)	6·46 (−3)	4·19 (−3)	2·41 (−3)	5·80 (−4)

	$Nu_0^{(1)}$, $a = 100$				$Nu_1^{(1)}$, $a = 100$			
M	$\lambda = 2$	$\lambda = 1$	$\lambda = 0$	$\lambda = -1$	$\lambda = 2$	$\lambda = 1$	$\lambda = 0$	$\lambda = -1$
0	5·50	5·09	4·70	2·16	5·62	5·09	4·59	2·06
2	5·13	4·74	4·37	2·01	5·23	4·74	4·28	1·92
4	4·32	3·98	3·66	1·68	4·39	3·98	3·60	1·62
10	2·40	2·19	2·00	9·08 (−1)	2·42	2·19	1·98	8·91 (−1)
50	4·95 (−1)	4·48 (−1)	4·04 (−1)	1·81 (−1)	4·96 (−1)	4·48 (−1)	4·02 (−1)	1·80 (−1)
100	2·45 (−1)	2·21 (−1)	1·99 (−1)	8·88 (−2)	2·45 (−1)	2·21 (−1)	1·99 (−1)	8·86 (−2)
150	1·63 (−1)	1·47 (−1)	1·32 (−1)	5·88 (−2)	1·63 (−1)	1·47 (−1)	1·32 (−1)	5·87 (−2)
200	1·22 (−1)	1·10 (−1)	9·85 (−2)	4·39 (−2)	1·30 (−1)	1·10 (−1)	1·08 (−1)	4·91 (−2)

increased and the heat transfer at the wall was greatly changed by this effect.

However when K, α and λ remain fixed and M increases the main feature is a reduction in magnitude of the velocity and temperature. The reason for this is as follows. An increase in M, i.e. the applied magnetic field strength causes greater interaction between the fluid motion and the magnetic field and hence an increase in the Lorentz force. Since this force opposes the buoyancy force the velocity will be decreased leading to a reduction in the viscous and Joulean dissipation and so a reduction in the temperature. The influence of large magnetic field on the net mass flow and heat transfer coefficients is readily seen from Tables 3–4. Thus if $\alpha = 100$, $\lambda = 1$, $K = 10$ and $M = 200$ then $\bar{q}$ is reduced to 2 per cent and Nu to 50 per cent of the magnetic field free case. In particular as $M \to \infty$ then $\bar{q} \to 0$ and the temperature profile tends to the ordinary conduction profile. Note also that when $\lambda = 1$, i.e. equal wall temperatures the velocity profile, evaluated neglecting dissipation effects, is similar to that in the Hartmann–Lazarus flow. Moreover as in the Hartmann–Lazarus flow a fluid boundary layer develops near the wall as M increases, and the velocity is then virtually constant across the gap and varies rapidly to zero near the walls.

It remains now to apply the above results to liquid metals such as mercury and liquid sodium. As an example the following natural convection flow configuration is chosen with mercury as the fluid. The plates, maintained at constant temperature 25°C, are taken to be 1 cm apart and the hydrostatic temperature to be 20°C. Thus $\alpha = 0$, $\lambda = 1$, $a = 1$ and $\theta_w = 5$°C. From Table 1 we obtain $P = 2{\cdot}68\ 10^{-2}$, $K_A = 1{\cdot}28\ 10^{-7}$, $G = 6{\cdot}99\ 10^5$ and the applied magnetic flux density $B_0 = 38{\cdot}4\ M$ gauss. Since $K = PG\ K_A = 2{\cdot}4\ 10^{-3}$ dissipation effects will be negligible and so the temperature across the gap will be constant and equal to 25°C. From Table 4 we may obtain the actual average velocity in cm/sec for various B_0. Hence if $B_0 = 0$, $\bar{u} = 66$ cm/sec; if $B_0 = 76{\cdot}8$ gauss, $\bar{u} = 62$ cm/sec; if $B_0 = 3{\cdot}84\ 10^2$ gauss, $\bar{u} = 32$ cm/sec; if $B_0 = 1{\cdot}54\ 10^3$ gauss, $\bar{u} = 9$ cm/sec and finally if $B_0 = 7{\cdot}68\ 10^3$ gauss, $\bar{u} = 2$ cm/sec. These results imply imply that for mercury a field of order 10^4 gauss is necessary to reduce the flow rate to one per cent of that in the magnetic field free case. When liquid sodium is used as the fluid the field necessary for the same reduction is not quite so large, as shown by the following example. We consider the flow configuration for which $T_s = 200$°C, $\theta_w = 50$°C, $\lambda = 1$ and $a = 1$. From Table 1 $P = 7{\cdot}38\ 10^{-3}$, $K_A = 1{\cdot}61\ 10^{-8}$, $G = 4{\cdot}37\ 10^5$, $K = 5{\cdot}295\ 10^{-5}$ and $B_0 = 7{\cdot}84\ M$ gauss. Again from Table 4 if $B_0 = 0$, $\bar{u} = 181$ cm/sec; if $B_0 = 15{\cdot}7$ gauss, $\bar{u} = 170$ cm/sec; if $B_0 = 78{\cdot}4$ gauss, $\bar{u} = 109$ cm/sec; if $B_0 = 3{\cdot}14\ 10^2$ gauss, $\bar{u} = 25{\cdot}8$ cm/sec. Furthermore as in the previous example for mercury heat is transferred across the gap by conduction alone, giving $T = 250$°C for $0 \leqslant Y \leqslant 1$.

The induced magnetic flux density B_x is not without interest. Consider the magnitude of B_x for the mercury flow configuration cited above. From Table 1 the magnetic viscosity $\chi = (1/4\pi\sigma\mu_e\nu) = 6{\cdot}74\ 10^6$ and so the magnetic Grashof number $G_M = 0{\cdot}104$. If we choose $M = 10$ then the applied magnetic flux density $B_0 = 384$ gauss; from Table 4 if $\alpha = 0$, $\lambda = 1$, $M = 10$ then $\bar{u}_0 = 4\ 10^{-2}$, also $\int_0^Y U_0 \mathrm{d}Y$ may be evaluated by numerical integration at $Y = 0(0{\cdot}2)\ 1{\cdot}0$ using the values of U_0 given in Table 2. Since $K = 2{\cdot}4\ 10^{-3}$ then from equation (2·32) we obtain that $B_x = 0$, $0{\cdot}12$, $0{\cdot}04$, 0, $-0{\cdot}04$, $-0{\cdot}12$, 0 gauss at $Y = 0(0{\cdot}1)1$ cm. The magnitude of the induced field is thus small in comparison with the applied magnetic field for this particular flow configuration. Moreover in this example B_x is small in comparison with the earth's magnetic field for which $B_x^{(E)} = 0{\cdot}44$ gauss and $B_0^{(E)} = 0{\cdot}17$ gauss approximately in the British Isles.

3. EXAMPLE II: ON THE TWO-DIMENSIONAL NATURAL CONVECTION FLOW DUE TO AN ELECTRIC CURRENT IN A HORIZONTAL CIRCULAR TUBE FILLED WITH AN ELECTRICALLY CONDUCTING VISCOUS FLUID

3(a) *Statement of problem and governing equations*

The special example to be considered is as follows: A long thin walled circular tube (made of non-conducting material) is filled with electrically conducting viscous fluid and placed with its axis in a horizontal position. A direct current of mean density J_0 per unit of area flows axially through the tube whose outer surface is maintained at constant temperature T_s, i.e. the temperature of the coolant. Moreover the coolant is assumed to be non-magnetic and a non-conductor. The equations describing the resulting steady two-dimensional natural convection flow for the central portion of the tube (i.e. at sufficient distance from either electrodes at the ends of the tube) are given by equations (1.10) to (1.16) when $Q = 0$.

It is convenient to use cylindrical polar co-ordinates r, ϕ and z. The z axis is taken to be the axis of the cylinder and the angular co-ordinate ϕ is measured from a vertical plane through this axis; the velocity components are denoted by u_r and u_ϕ and the magnetic field components by H_r and H_ϕ. The basic equations (1.10) to (1.16) transformed to cylindrical polar co-ordinates give for the fluid or region (1) the set of equations:

$$\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_\phi}{\partial \phi} = 0, \tag{3.1}$$

$$\left.\begin{aligned} u_r \frac{\partial u_r}{\partial r} + \frac{u_\phi}{r}\frac{\partial u_r}{\partial \phi} - \frac{u_\phi^2}{r} = &-\frac{1}{\rho}\frac{\partial p_d}{\partial r} \\ &+ \nu\left(\nabla^2 u_r - \frac{u_r}{r^2} + \frac{2}{r^2}\frac{\partial u_\phi}{\partial \phi}\right) \\ &+ \beta g(T - T_s)\cos\phi - \frac{\mu_e}{\rho} J_z H_\phi, \end{aligned}\right\} \tag{3.2}$$

$$\left.\begin{aligned} u_r \frac{\partial u_\phi}{\partial r} + \frac{u_\phi}{r}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r u_\phi}{r} = &-\frac{1}{\rho}\frac{1}{r}\frac{\partial p_d}{\partial \phi} \\ &+\nu\left(\nabla^2 u_\phi - \frac{u_\phi}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \phi}\right) \\ &- \beta g(T - T_s)\sin\phi + \frac{\mu_e}{\rho} J_z H_r, \end{aligned}\right\} \tag{3.3}$$

$$\frac{\partial}{\partial r}(r H_r) + \frac{\partial H_\phi}{\partial \phi} = 0, \tag{3.4}$$

$$\begin{aligned} J_z &= \frac{1}{4\pi}\left(\frac{1}{r}\frac{\partial}{\partial r}(r H_\phi) - \frac{1}{r}\frac{\partial H_r}{\partial \phi}\right) \\ &= \sigma\{E_0 + \mu_e(u_r H_\phi - u_\phi H_r)\}, \end{aligned} \tag{3.5}$$

and the energy equation is

$$\rho c_p\left(u_r \frac{\partial T}{\partial r} + \frac{u_\phi}{r}\frac{\partial T}{\partial \phi}\right) = k\nabla^2 T + \frac{J_z^2}{\sigma} + \Phi', \tag{3.6}$$

where $$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}.$$

The viscous dissipation function Φ' takes the form

$$\begin{aligned} \Phi' = 2\left(\frac{\partial u_r}{\partial r}\right)^2 + 2\left(\frac{1}{r}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r}\right)^2 \\ + \left(\frac{1}{r}\frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}\right)^2. \end{aligned} \tag{3.7}$$

For the coolant or region (2) the relevant equations are:

$$\left.\begin{aligned} &\frac{1}{r}\frac{\partial}{\partial r}(r H_\phi) - \frac{1}{r}\frac{\partial H_r}{\partial \phi} = 0 \\ \text{and} \quad &\frac{\partial}{\partial r}(r H_r) + \frac{\partial H_\phi}{\partial \phi} = 0 \end{aligned}\right\} \tag{3.8}$$

The boundary conditions are:

$$\left.\begin{aligned} &u_r = u_\phi = 0,\ T = T_s,\ \mu_{1e}H_{1r} = \mu_{2e}H_{2r}, \\ &H_{1\phi} = H_{2\phi} \text{ on } r = a,\ u_r,\ u_\phi,\ T,\ H_r \text{ and} \\ &H_\phi \text{ are finite at } r = 0, \\ &\text{also } H_{2r} \text{ and } H_{2\phi} \to 0 \text{ as } r \to \infty, \end{aligned}\right\} \tag{3.9}$$

where a is the radius of the tube and the suffices refer to the fluid and coolant respectively.

In formulating the boundary conditions (3.9) the thickness of the non-conducting tube wall is assumed to be small in comparison with the radius of the tube and thus there is little variation in H_r and H_ϕ in the wall material and so the usual condition on **H** at the interface can be taken as between fluid and coolant. Furthermore at the interface there can be no normal flow of current, i.e. $J_r = 0$, a condition which is satisfied by this type of two-dimensional solution; also at the interface the tangential component of $E = E_0$ must be continuous implying that in the coolant there is a constant electric field $E_z = E_0$. Note that these boundary conditions on **H** and **E** are identical with those for the Joule heating, by a direct current, of an infinitely long cylindrical wire placed in a non-conducting medium.

On choosing J_0, $4\pi a J_0$, $(J_0^2 a^2/k\sigma)$ and ν/a as units of current, field strength, temperature and velocity respectively then the steady two-dimensional flow may be shown to depend on the following parameters: $\chi = (1/4\pi\sigma\mu_e)$, the magnetic viscosity; $G = (\beta g a^5 J_0^2/k\sigma\nu^2)$, the Grashof number; $P = (\rho c_p \nu/k)$, the Prandtl number; and the dimensionless group $K_A = (\beta g a/c_p)$. Equations (3.1) to (3.9) are thus reduced to non-dimensional form using the following dimensionless independent and dependent variables:

$$\left.\begin{aligned} R = r/a,\ U_R = \frac{au_r}{\nu K},\ U_\phi = \frac{au_\phi}{\nu K}, \\ h_R = \frac{H_r}{4\pi a J_0},\ h_\phi = \frac{H_\phi}{4\pi a J_0}, \\ E_0' = \frac{\sigma E_0}{J_0} \text{ and } \Theta = (T - T_s)\frac{k\sigma}{J_0^2 a^2}, \end{aligned}\right\} \quad (3.10)$$

whereas in section (2) $K = PGK_A$.

In region (1) the stream function Ψ and the magnetic vector potential $\mathbf{A} = (0, 0, A)$ are defined by

$$RU_R = \frac{\partial\Psi}{\partial\phi},\ U_\phi = -\frac{\partial\Psi}{\partial R},\ Rh_R = \frac{\partial A}{\partial\phi},$$
$$h_\phi = -\frac{\partial A}{\partial R} \quad (3.11)$$

respectively, then on eliminating the pressure p_d the governing equations become:

$$\nabla^4\Psi + K\frac{\partial(\Psi, \nabla^2\Psi)}{\partial(R, \phi)} = \chi\epsilon^2\frac{\partial(A, \nabla^2 A)}{\partial(R, \phi)} - \chi\epsilon\frac{\partial(R\cos\phi, \Theta)}{\partial(R, \phi)}, \quad (3.12)$$

$$j_z = -\nabla^2 A = E_0' + \frac{K}{\chi}\frac{\partial(\Psi, A)}{\partial(R, \phi)}, \quad (3.13)$$

$$\nabla^2\Theta + (\nabla^2 A)^2 + \frac{K}{\epsilon^2\chi^2}\Phi' + PK\frac{\partial(\Psi, \Theta)}{\partial(R, \phi)} = 0, \quad (3.14)$$

where $\dfrac{\partial(E, F)}{\partial(R, \phi)} = \dfrac{\partial E}{\partial R}\dfrac{1}{R}\dfrac{\partial F}{\partial\phi} - \dfrac{\partial E}{R\partial\phi}\dfrac{\partial F}{\partial R}$,

and $\epsilon = 1/PK_A\chi$ is a convenient dimensionless group. For the coolant or region (2) the magnetic field is determined by

$$h_R = \frac{\partial\Phi}{\partial R} \text{ and } h_\phi = \frac{1}{R}\frac{\partial\Phi}{\partial\phi}. \quad (3.15)$$

where the scalar potential Φ satisfies Laplaces equation

$$\nabla^2\Phi = 0. \quad (3.16)$$

The boundary conditions become

$$\Phi,\ \Theta,\ A \text{ finite at } R = 0,$$
$$\Psi = \frac{\partial\Psi}{\partial R} = \Theta = 0 \text{ at } R = 1.$$
$$\gamma\frac{\partial A}{\partial\phi} = \frac{\partial\Phi}{\partial R},\ \frac{\partial A}{\partial R} = -\frac{\partial\Phi}{\partial\phi} \text{ at } R = 1. \quad (3.17)$$

and $$\frac{1}{R}\frac{\partial\Phi}{\partial\phi} \to 0,\ \frac{\partial\Phi}{\partial R} \to 0 \text{ as } R \to \infty.$$

where $\gamma = \mu_{1e}/\mu_{2e}$ is the ratio of the magnetic permeability of the fluid to that of the coolant.

The above equations (3.12) to (3.17) have a non-zero solution provided the dimensionless electric field E_0' is non-zero. Since the velocity and magnetic field components are independent of z then equation (3.5) implies that $\iint_s J_z dr d\phi$ is constant and equal to the net flow of current crossing any tube cross-section in the z-direction. If this is chosen such that $\iint_s J_z r dr d\phi = \pi a^2 J_0$, then on integrating equation (3.13) over the tube cross-section and using Gauss's theorem together with the boundary condition that $\Psi = 0$ at $R = 1$, we obtain $E_0' = 1$. Thus the dimensionless form of Ohm's law is

$$j_z = -\nabla^2 A = 1 + \frac{K}{\chi}\frac{\partial(\Psi, A)}{\partial(R, \phi)}. \quad (3.18)$$

Consider now the order of magnitude of the parameter K, χ and ϵ relevant to liquid metals such as mercury and liquid sodium. If the radius of the tube is a cm and the applied current is i_0 amp/cm² then from Table 1 we obtain for mercury ($T_s = 20°$C): $P = 2{\cdot}68\ 10^{-2}$, $K_A = 1{\cdot}28\ 10^{-7}a$, $\chi = 6{\cdot}74\ 10^6$, $\epsilon = 43{\cdot}3/a$, $G = 1{\cdot}69\ 10^2 a^5 i_0^2$ and $K = 5{\cdot}8\ 10^{-7} a^6 i_0^2$; and for liquid sodium ($T_s = 200°$C): $P = 7{\cdot}38\ 10^{-3}$, $K_A = 1{\cdot}61\ 10^{-8}a$, $\chi = 2{\cdot}18\ 10^5$, $\epsilon = 3{\cdot}85 10^4/a$, and $K = 1{\cdot}74\ 10^{-11} a^6 i_0^2$. Thus in general if $a = 0(1)$ and i_0 is small or moderately large then $K \ll 1$, $\chi > \epsilon > 1$, and χ and ϵ^2 may be of the same order of magnitude. In fact, as in Example I, the magnitude of K controls the viscous and Joulean dissipation. The next section deals with series expansions in powers of K for Ψ, Θ, A and Φ.

3(b) *Series expansion for small K*

We assume expansions of the type

$$\left.\begin{aligned}\Psi &= \Psi_0 + K\Psi_1 + \ldots,\\ A &= A_0 + KA_1 + \ldots,\\ \Theta &= \Theta_0 + K\Theta_1 + K^2\Theta_2 + \ldots,\\ \text{and } \Phi &= \Phi_0 + K\Phi_1 + K^2\Phi_2 + \ldots\end{aligned}\right\} \quad (3.19)$$

On substituting the equations (3.19) into (3.12), (3.14), and (3.18) and equating coefficients of like powers of K there results:

Zero-Order functions

$$\nabla^2 A_0 + 1 = 0, \quad (3.20)$$

$$\nabla^2 \Phi_0 = 0, \quad (3.21)$$

$$\nabla^2 \Theta_0 + 1 = 0, \quad (3.22)$$

$$\nabla^4 \Psi_0 + \chi\epsilon \frac{\partial(R\cos\phi, \Theta_0)}{\partial(R, \phi)} = 0; \quad (3.23)$$

First-Order functions

$$\nabla^2 A_1 + \frac{1}{\chi}\frac{\partial(\Psi_0, A_0)}{\partial(R, \phi)} = 0, \quad (3.24)$$

$$\nabla^2 \Phi_1 = 0, \quad (3.25)$$

$$\nabla^2\Theta_1 + 2\nabla^2 A_0 \nabla^2 A_1 + \frac{1}{\epsilon^2\chi^2}\Phi_0' + P\frac{\partial(\Psi_0, \Theta_0)}{\partial(R, \phi)} = 0, \quad (3.26)$$

$$\nabla^4\Psi_1 + \frac{\partial(\Psi_0, \nabla^2\Psi_0)}{\partial(R, \phi)} + \chi\epsilon\frac{\partial(R\cos\phi, \Theta_1)}{\partial(R, \phi)} - \chi\epsilon^2\left\{\frac{\partial(A_1, \nabla^2 A_D)}{\partial(R, \phi)} + \frac{\partial(A_0, \nabla^2 A_1)}{\partial(R, \phi)}\right\}; \quad (3.27)$$

Second-Order functions

$$\nabla^2 A_2 + \frac{1}{\chi}\left\{\frac{\partial(\Psi_0, A_1)}{\partial(R, \phi)} + \frac{\partial(\Psi_1, A_0)}{\partial(R, \phi)}\right\} = 0. \quad (3.28)$$

$$\nabla^2\Phi_2 = 0, \quad (3.29)$$

$$\nabla^2\Theta_2 + (\nabla^2 A_1)^2 + 2\nabla^2 A_0\nabla^2 A_2 + \frac{1}{\chi^2\epsilon^2}\Phi_1' + P\left\{\frac{\partial(\Psi_0, \Theta_1)}{\partial(R, \phi)} + \frac{\partial(\Psi_1, \Theta_0)}{\partial(R, \phi)}\right\}. \quad (3.30)$$

In equations (3.26) and (3.30) Φ_0' and Φ_1' are the zeroth and first-order viscous dissipation functions, which can be obtained using (3.7), (3.10), (3.11) and (3.17). The boundary conditions for A_n, Ψ_n, Θ_n and Φ_n for $n = 0$, 1 and 2 are as follows:

$$\left.\begin{aligned}&A_n, \Theta_n, \Psi_n \text{ are finite at } R = 0,\\ &\Psi_n = \frac{\partial\Psi_n}{\partial R} = \Theta_n = 0, \frac{\partial A_n}{\partial\phi} = \frac{\partial\Phi_n}{\partial R},\\ &\qquad\frac{\partial A_n}{\partial R} = -\frac{\partial\Phi_n}{\partial\phi} \text{ at } R = 1,\\ &\text{and } \frac{1}{R}\frac{\partial\Phi_n}{\partial\phi} \to 0, \frac{\partial\Phi_n}{\partial R} \to 0 \text{ as } R \to \infty.\end{aligned}\right\} \quad (3.31)$$

Here we have assumed that the fluid and coolant are non-magnetic, i.e. $\gamma = \mu_{1e}/\mu_{2e} = 1$.

The equations (3.20) to (3.31) are readily solved and higher approximations to A, Ψ, Θ and Φ could be obtained. However the incidental arithmetic involved in obtaining even the second-order functions indicates the limited usefulness of the series expansion (3.19). In the following we shall make use of the fact that $\chi > \epsilon > 1$ and $\chi = 0(\epsilon^2)$ if $a = 0(1)$ and thus give only the important terms in the rather lengthy expressions to be derived for A_n, Ψ_n, Θ_n and Φ_n.

Consider first the zeroth-order functions defined by equations (3.20) to (3.23) and (3.31). These are:

$$\left.\begin{aligned}&A_0 = -\tfrac{1}{4}(R^2 + a_0),\ \Phi_0 = \tfrac{1}{2}\phi + b_0,\\ &\Theta_0 = \tfrac{1}{4}(1 - R^2),\\ &\qquad\Psi_0 = \frac{\chi\epsilon}{384}(R^5 - 2R^3 + R)\sin\phi,\end{aligned}\right\} \quad (3.32)$$

where a_0 and b_0 are constants of integration. Hence in the fluid

$$h_R^{(0)} = 0,\ h_\phi^{(0)} = R/2,\ j_z^{(0)} = 1, \quad (3.33)$$

$$U_R^{(0)} = \frac{\chi\epsilon}{384}(R^4 - 2R^2 + 1)\cos\phi,$$

$$U_\phi^{(0)} = -\frac{\chi\epsilon}{384}(5R^4 - 6R^2 + 1)\sin\phi, \quad (3.34)$$

and for the coolant

$$h_R^{(0)} = 0,\ h_\phi^{(0)} = \frac{1}{2R}. \quad (3.35)$$

The first-order functions refine the above approximation by taking into account (a) the interaction of the fluid motion described by expression (3.34) and the magnetic field given by expression (3.33), and (b) the effect of viscous and Joulean dissipation. The solution of equations (3.24) to (3.27) subject to the boundary conditions (3.38) is:

$$\left.\begin{aligned}
A_1 &= \frac{-\epsilon}{96 \times 384}(R^7 - 4R^5 + 6R^3 - 4R)\cos\phi + a_1,\ \Phi_1 = \frac{\epsilon}{96\times 384}\frac{\sin\phi}{R} + b_1,\\
\Theta_1 &= -\frac{1}{96\times 384}\{\epsilon(2+\chi P)(R^7 - 4R^5 + 6R^3 - 3R)\cos\phi + 0(1)\},\\
\Psi_1 &= \frac{-\chi\epsilon}{240(384)^2}\{\chi\epsilon(R^{10} - 4R^8 + 5R^6 - 2R^4)\sin 2\phi - \tfrac{1}{2}(2+\chi P)\epsilon(R^{10} - 8R^8\\
&\quad + 30R^6 - 40R^4 + 17R^2)\cos 2\phi + 2\epsilon^2(3R^9 - 20R^7 + 60R^5 - 72R^3 + 29R)\sin\phi + 0(1)\},
\end{aligned}\right\} \quad (3.36)$$

where a_1 and b_1 are constants of integration. Thus in the fluid

$$\left.\begin{aligned}
h_R^{(1)} &= \frac{\epsilon}{96\times 384}(R^6 - 4R^4 + 6R^2 - 4)\sin\phi,\ h_\phi^{(1)} = \frac{\epsilon}{96\times 384}(7R^6 - 20R^4\\
&\quad + 18R^2 - 4)\cos\phi,\ j_z^{(1)} = \frac{\epsilon}{768}(R^5 - 2R^3 + R)\cos\phi,
\end{aligned}\right\} \quad (3.37)$$

$$\left.\begin{aligned}
U_R^{(1)} &= \frac{-\chi\epsilon}{(384)^2 240}\{2\chi\epsilon(R^9 - 4R^7 + 5R^5 - 2R^3)\cos 2\phi - \epsilon(2+\chi P)(R^9 - 8R^7\\
&\quad + 30R^5 - 40R^3 + 17R)\cos 2\phi + 2\epsilon^2(3R^8 - 20R^6 + 60R^4 - 72R^2 + 29)\cos\phi + 0(1)\},\\
U_\phi^{(1)} &= \frac{\chi\epsilon}{(384)^2 240}\{\chi\epsilon(10R^9 - 32R^7 + 30R^5 - 8R^3)\sin 2\phi - \epsilon(2+\chi P)(10R^9 - 64R^7\\
&\quad + 180R^5 - 160R^3 + 34R)\sin 2\phi + 2\epsilon^2(27R^8 - 140R^6 + 300R^4 - 216R^2 + 29)\sin\phi + 0(1)\},
\end{aligned}\right\} \quad (3.38)$$

and for the coolant

$$h_R^{(1)} = \frac{-\epsilon}{96\times 384}\frac{\sin\phi}{R^2},\ h_\phi^{(1)} = \frac{\epsilon}{96\times 384}\frac{\cos\phi}{R^2}. \quad (3.39)$$

For the second order functions $j_z^{(2)}$, A_2 and Φ_2 we obtain:

$$\left.\begin{aligned}
j_z^{(2)} &= \frac{-\epsilon}{(384)^2 480}\{2\chi\epsilon(R^{10} - 4R^8 + 5R^6 - 2R^4)\cos 2\phi - \epsilon\chi P(R^{10} - 8R^8 + 30R^6\\
&\quad - 40R^4 + 17R^2)\cos 2\phi + 2\epsilon^2(3R^9 - 20R^7 + 60R^5 - 72R^3 + 29R)\cos\phi + 0(\epsilon)\},\\
A_2 &= \frac{\epsilon}{(384)^2 480}\left\{\frac{\chi\epsilon}{840}(12R^{12} - 70R^{10} + 140R^8 - 105R^6 + 28R^2)\cos 2\phi - \frac{\epsilon\chi P}{420}(3R^{12}\right.\\
&\quad - 35R^{10} + 210R^8 - 525R^6 + 595R^4 - 273R^2)\cos 2\phi + \frac{\epsilon^2}{20}(R^{11} - 10R^9 + 50R^7\\
&\quad \left. - 120R^5 + 145R^3 - 86R)\cos\phi + 0(\epsilon)\right\} + a_2,\\
&\text{and}\\
\Phi_2 &= \frac{\epsilon}{240(384)^2}\left\{\frac{5\epsilon\chi P}{42}\frac{\sin 2\phi}{R^2} + \frac{\chi\epsilon}{84}\frac{\sin 2\phi}{R^2} - 2\epsilon^2\frac{\sin\phi}{R} + 0(\epsilon)\right\} + b_2
\end{aligned}\right\} \quad (3.40)$$

where a_2 and b_2 are constants of integration.

It remains now to evaluate the second-order temperature function Θ_2 as defined by equations (3.30) and (3.31). The function Θ_2 can be written as

$$\Theta = \Theta_{2,0} + \Theta_{2,1} + \Theta_{2,2} \tag{3.41}$$

where the functions satisfy, respectively, the equations

$$\nabla^2\Theta_{2,0} + (\nabla^2 A_1)^2 + 2\nabla^2 A_0 \,.\, \nabla^2 A_2 = 0,$$

$$\nabla^2\Theta_{2,1} + \Phi_1' = 0,$$

and

$$\nabla^2\Theta_{2,2} + P\left(\frac{\partial(\Psi_0, \Theta_1)}{\partial(R, \phi)} + \frac{\partial(\Psi_1, \Theta_0)}{\partial(R, \phi)}\right) = 0.$$

The functions $\Theta_{2,0}$, $\Theta_{2,1}$ and $\Theta_{2,2}$ are the temperature distributions due to Joulean dissipation, viscous dissipation and heat transferred by convection, respectively. The boundary conditions are that $\Theta_{2,p}$ is finite at $R = 0$ and $\Theta_{2,p}(1, \phi) = 0$ for $p = 0, 1$ and 2 respectively.

The appropriate solutions of these equations are:

$$\left.\begin{aligned}
\Theta_{2,0} &= \frac{-\epsilon^2}{240(384)^2}\left\{\frac{\chi\epsilon}{840}(12R^{12} - 70R^{10} + 140R^8 - 105R^6 + 23R^2)\cos 2\phi - \frac{\epsilon\chi P}{420}(3R^{12}\right.\\
&\quad - 35R^{10} + 210R^8 - 525R^6 + 595R^4 - 248R^2)\cos 2\phi + \frac{\epsilon^2}{20}(R^{11} - 10R^9 + 50R^7\\
&\quad \left. - 120R^5 + 145R^3 - 66R)\cos\phi + 0(\epsilon)\right\},\\
\Theta_{2,1} &= \frac{1}{15(384)^3}\left\{\frac{\chi\epsilon}{1680}(693\,R^{13} - 2640\,R^{11} + 3220\,R^9 - 1344\,R^7 + 71\,R^3)\cos 3\phi\right.\\
&\quad - \frac{\epsilon\chi P}{3360}(693\,R^{13} - 4920R^{11} + 14840R^9 - 15120R^7 + 3750R^5 + 937R^3)\cos 3\phi + \frac{2\epsilon^2}{5}\\
&\quad \times (6R^{12} - 35R^{10} + 80R^8 - 60R^6 + 9R^2)\cos 2\phi - \frac{\chi\epsilon}{280}(170R^{13} - 756R^{11} + 1225R^9\\
&\quad - 910R^7 + 280R^5 - 9R)\cos\phi - \frac{\epsilon^2}{10}(30R^{12} - 192R^{10} + 525R^8 - 640R^6\\
&\quad \left. + 360R^4 - 83) + 0(\epsilon)\right\},\\
\Theta_{2,2} &= \frac{P\chi\epsilon}{240(384)^2}\left\{\frac{\chi\epsilon}{840}(12R^{12} - 70R^{10} + 140R^8 - 105R^6 + 23R^2)\cos 2\phi - \frac{\epsilon\chi P}{1680}(72R^{12}\right.\\
&\quad - 490R^{10} + 1540R^8 - 2625R^6 + 2380R^4 - 877R^2)\cos 2\phi + \frac{\epsilon^2}{20}(R^{11} - 10R^9 + 50R^7\\
&\quad - 120R^5 + 145R^3 - 66R)\cos\phi - \frac{\epsilon\chi P}{144}(30R^{12} - 216R^{10} + 675R^8 - 1140R^6 + 1080R^4\\
&\quad \left. - 540R^2 + 111) + 0(1)\right\}
\end{aligned}\right\} \tag{3.42}$$

In the next section the physical implications of these results are discussed.

3(c) *Results and Discussion*

Consider now the flow configuration as predicted by section 3(b) for small K. The zeroth-order approximation given by expressions (3.36) to (3.39) implies that the temperature distribution and magnetic field components are identical in form with those occurring in the Joule heating of an infinitely long solid cylinder with the same external conditions. Temperature gradients now exist in the fluid producing the buoyancy force resulting in the formation of a non-uniform fluid motion having convective cells on either side of the vertical plane $\phi = 0, \pi$ (see expression (3.34)). The fluid rises along the vertical plane and flows downward along the cool wall on either side of the vertical plane; also the fluid is stationary at $[1/(\surd 5), \pi/2]$ and $[1/(\surd 5), (3\pi)/2]$. The vertical flow along $\phi = 0, \pi$ now distorts the magnetic lines of force, given by expression (3.37), in the vertical direction. Thus we might expect an increase in local current density along the plane $\phi = 0$. This is borne out by the first approximation to j_z (see expression (3.37)) where $j_z^{(1)}$ has a maximum at $[1/(\surd 5), 0]$. Along $\phi = \pi$ the vertical flow distorts the magnetic lines of force away from the tube surface thus producing a maximum decrease in local current density at $[1/(\surd 5), \pi]$.

At this stage in the analysis the current density is no longer uniform over the tube cross-section and thus several modifications to the flow and temperature are necessary. Thus expression (3.36) for Θ_1 implies a maximum increase at (0·45, 0) and a maximum decrease at (0·46. π) coinciding approximately with the maximum increase and decrease for $j_z^{(1)}$ at $[1/(\surd 5), 0]$ and $[1/(\surd 5), \pi]$ respectively. A Lorentz force now exists opposing the buoyancy force and thus reduces in magnitude the velocity components. This is evident from the first-order velocity components as given by expressions (3.38). These components also imply that the "regions" of stationary fluid are now to the right of $[1/(\surd 5), \pi/2]$ and to the left of $[1/(\surd 5), (3\pi)/2]$. Finally as a consequence of the non-uniformity of current density there is an increase in the ϕ-component of the magnetic field in the neighbourhood of the axis $\phi = 0$ and a decrease in the neighbourhood of $\phi = \pi$ for both fluid and coolant [see expressions (3.37) and (3.39)]; furthermore a radial magnetic field is set up which acts radially inward for the coolant and fluid if $0 < \phi < \pi$ and radially outwards for $\pi < \phi < 2\pi$.

The second-order approximation adds little information to the above general trends. However, provided K is small there will be an overall small increase in j along the vertical axis and a slight decrease in the neighbourhood of the stationary points. A slight decrease in the local temperature will also occur in this neighbourhood [see expression (3.42)].

The above discussion gives some insight into the type of flow which may occur for large values of K. Thus we might expect isothermal cores on either side of the vertical plane. The fluid will rise in a narrow jet up the vertical plane and return down the wall on either side in a thin boundary layer which acts as a shield between the isothermal core and the wall. Furthermore there should be an increase in the local current density and temperature in the neighbourhood of (0·5, 0) and a corresponding decrease at (0·5, π).

Let us consider some thermal characteristics of the flow configuration when K is small. An average wall Nusselt number may be defined as

$$\overline{Nu} = \frac{a\int_0^{2\pi} (\mathrm{d}T/\mathrm{d}r)_{r=a}\, \mathrm{d}\phi}{2\pi(T_c - T_s)} \tag{3.43}$$

where $T_c - T_s = (J_0^2\, a^2/4K\sigma)$ is the temperature difference between the axis of the tube and the coolant when the thermal energy is transferred by conduction alone. Using the various transformations given by equation (3.10) and the complete expressions for the Θ_n as derived in section 3(b) we obtain

$$\overline{Nu} = 2 + \frac{1}{(96)^2} K + \frac{2}{5(384)^3}(8\epsilon^3 - 16\epsilon^2 + 1)K^2 + 0(K^3). \tag{3.44}$$

The axial temperature T_0 due to Joule heating and as modified by the non-uniform convection flow was found to be

$$T_0 = T_s + \frac{J_0^2 a^2}{4K\sigma}\left[1 + \frac{14}{3(384)^2} K + \frac{2}{75(384)^3}\left\{372\epsilon^3 + (83 + 350\chi P + 175\chi^2 P^2)\epsilon^2 + \frac{737}{168}\right\} K^2 + 0(K^3)\right] \quad (3.45)$$

In Table 6 the actual temperature difference $(T_0 - T_s)$, as calculated from expression (3.45), is compared with the fictitious temperature difference $(T_c - T_s) = (J_0^2 a^2)/(K\sigma)$. The fluids taken were mercury and liquid sodium with coolant temperatures 20° and 200°C respectively; the diameter of the tube is 1 cm and the current density is i_0 amp/cm². We note that for a tube of 1 cm diameter expression (3.45) is probably accurate for mercury if $i_0 \leqslant 80$ amp/cm² and for liquid sodium if $i_0 \leqslant 510^3$ amp/cm². Moreover within these current density limitations the average wall Nusselt number is 2 for both fluids [see expression (3.44)]. In view of this, and the small temperature differences existing between the fluid and the coolant when extremely large currents are applied, it seems unlikely that these results could be verified experimentally.

Table 6

Mercury (T_s = 20°C, dia. 1 cm)			Liquid sodium (T_s = 200°C, dia. 1 cm)		
i_0	$T_c - T_s$	$T_0 - T_s$	i_0	$T_c - T_s$	$T_0 - T_s$
40	0·12	0·12	1000	1·05	1·05
50	0·19	0·18	2000	4·20	4·18
60	0·27	0·25	3000	9·45	9·17
70	0·37	0·31	4000	16·8	15·2
80	0·48	0·35	5000	26·3	20·2

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